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ON RECIPROCAL METHODS IN THE DIFFERENTIAL CALCULUS.

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(*Continued from* Vol. vii. p. 166.)

We pass now to the case in which the number of quantities in the two sets corresponding to parameters and coordinates is not the same.

(1). Let the two sets of connected quantities be $x_1, x_2 \ldots x_n$ and $a_1, a_2 \ldots a_m$, and let m be greater than n. The following theorem may then be established.

Theorem II. If two sets of quantities $x_1, x_2 \ldots x_n$ and $a_1, a_2 \ldots a_m$, m being greater than n, are connected by any relations $u_1 = 0$, $u_2 = 0, \ldots u_r = 0$, and if the former set varying in subjection to a certain condition among themselves $X = 0$, establish among the other set $a_1, a_2 \ldots a_m$, with which they are connected, a relation $A = 0$; then the set $a_1, a_2 \ldots a_n$, varying either in subjection to the single condition $A = 0$, or to that condition together with any other $m - n$ arbitrary conditions among themselves, will establish among the set $x_1, x_2 \ldots x_n$ the relation $X = 0$.

Let us suppose the quantities $a_1, a_2 \ldots a_m$ connected with another set of quantities $b_1, b_2 \ldots b_m$ by m arbitrary functional relations, as for example,

$$b_1 = f_1(a_1, a_2 \ldots a_m),\ b_2 = f_2(a_1, a_2 \ldots a_m), \ldots b_m = f_m(a_1, a_2 \ldots a_m) \ldots (21).$$

By virtue of these relations we can transform $u_1, u_2 \ldots u_r$ into functions of $x_1, x_2 \ldots x_n$, $b_1, b_2 \ldots b_m$.

If we do this, and cause $x_1, x_2 \ldots x_n$ to vary in subjection to the condition $X = 0$, we shall establish among the constants $b_1, b_2 \ldots b_m$ a relation which may be represented by $B = 0$. And it appears from the general Theorem I., that if we regard in B any n of the quantities $b_1, b_2 \ldots b_n$ as

parameters, and cause them to vary in subjection to the condition $B = 0$, the remaining $m - n$ of those quantities remaining constant, we shall establish among the quantities $x_1, x_2 \dots x_n$ the relation $X = 0$. Or, which amounts to the same thing, we may suppose the whole series of quantities $b_1, b_2 \dots b_m$ to vary in subjection to the conditions

$$B = 0,$$

$$b_{n+1} = \text{const.}, \quad b_{n+2} = \text{const.}, \dots b_m = \text{const.} \dots\dots (22),$$

and the resulting relation among $x_1, x_2 \dots x_n$ will establish the condition $X = 0$.

Now it is obviously indifferent, whether in the above process we make $b_1, b_2 \dots b_m$ vary independently in subjection to the above conditions, or whether we write for $b_1, b_2 \dots b_m$ their values in terms of $a_1, a_2 \dots a_m$ in the several equations, and then make $a_1, a_2 \dots a_m$ vary in subjection to the conditions consequent upon this change; just as in seeking the maximum value of a function of certain quantities, it is indifferent whether we equate to 0 the differentials of these quantities, or transform them by functional relations into another system of quantities, and equate to 0 the differentials taken with respect to the latter. If we effect, then, the proposed change, B will obviously become A, and the equations (22) will become

$$f_{n+1}(a_1, a_2 \dots a_m) = \text{const.}, \quad f_{n+2}(a_1, a_2 \dots a_m) = \text{const.},$$
$$\dots f_m(a_1, a_2 \dots a_m) = \text{const.} \dots\dots\dots\dots\dots (23).$$

Hence the variation of $a_1, a_2 \dots a_m$, in subjection to the condition $A = 0$ and to the $m - n$ arbitrary conditions above stated, will establish the condition $X = 0$.

As the conditions (23) are perfectly arbitrary, we may in any particular case entirely disregard them. Hence the conditions $A = 0$, when either n or any greater number of the quantities $a_1, a_2 \dots a_m$ are supposed to vary, is, with or without any other condition among those quantities, sufficient to establish the relation $X = 0$.

The most important application of this result to the inverse problem of envelopes is comprehended in the following Rule. We confine ourselves here to the case of surfaces.

To ascertain the conditions under which the surface whose equation is

$$\phi(x, y, z, a_1, a_2 \dots a_m) = 0 \dots\dots\dots\dots (24),$$

$a_1, a_2 \dots a_m$ being variable parameters, shall generate the envelope whose equation is

$$\chi(x, y, z) = 0 \dots\dots\dots\dots\dots\dots (25),$$

supposing $m > 3$.

RULE. Determine the equation of the envelope of the given surface regarding the coordinates x, y, z as parameters, subject to the condition (25), and the parameters $a_1, a_2 \ldots a_m$ as if they were coordinates. The resulting equation $A = 0$ will express the only *necessary* condition, but it may be associated according to choice with any number not exceeding $m - 3$ of arbitrary relations among the quantities $a_1, a_2 .. a_m$.

There is a remarkable circumstance attending the employment of the equation $A = 0$, which may serve as a verification of the method by which in any particular instance that equation has been determined. It is this: when we seek the envelope of ϕ subject to the single condition $A = 0$, we employ in the process a system of equations of m equations

$$\frac{d\phi}{da_1} + \lambda \frac{dA}{da_1} = 0, \quad \frac{d\phi}{da_2} + \lambda \frac{dA}{da_2} = 0, \ldots \frac{d\phi}{da_m} + \lambda \frac{dA}{da_m} = 0,$$

where for convenience ϕ stands for $\phi(x, y, z, a_1, a_2 \ldots a_m)$. On elimination of λ these give $m - 1$ equations connecting $x, y, z, a_1, a_2 \ldots a_m$. Taking into account the primitive equation (24), we have thus implicitly a system of m equations connecting these quantities, *i.e.* the two sets x, y, z, and $a_1, a_2 \ldots a_m$, together.

Now if the above equations were independent of each other and of the equation $A = 0$, they would furnish us, on elimination of x, y, z with $m - 3$, new equations connecting $a_1, a_2 \ldots a_m$. But in reality there is only one *necessary* relation connecting $a_1, a_2 \ldots a_m$, viz. the equation $A = 0$. Hence the m equations previously referred to are not independent of each other and of the equation $A = 0$, and in fact it will be found that the elimination of x, y, z will only reproduce the equation $A = 0$, verifying the observation that this is the only *necessary* condition among the constants $a_1, a_2 \ldots a_m$, and serving as a test of the correctness of the process by which that condition is determined. The above conclusion admits of formal proof. The following example will serve to illustrate both the rule and the accompanying remark.

Ex. Required the conditions under which the straight line whose equation is $ax + by = c$ shall have for its envelope the circle whose equation is $x^2 + y^2 = \frac{1-m}{m}$, a, b, and c being the variable parameters.

We have
$$ax + by = c \text{ ....} (26),$$
$$x^2 + y^2 = \frac{1-m}{m} \text{ ....} (27),$$
$$bx = ay \text{ ....} (28),$$
the last equation being got by eliminating dx and dy from the differentials of the two preceding ones.

Eliminating x and y from the above, we get
$$(m-1)(a^2+b^2) + mc^2 = 0 \text{ ....} (29).$$
Now if we seek the envelope of (26), subjecting the parameters a, b, c, to the above condition (29), then, whether we regard any two of the parameters as varying (according to the first case), or the three as varying (according to the second case), we obtain (27), the equation of the envelope. Let us confine ourselves to the latter case.

Our equations then are
$$ax + by - c = 0 \text{ ....} (30),$$
$$(m-1)(a^2+b^2) + mc^2 = 0 \text{ ....} (31),$$
whence, in the ordinary way, we get
$$\frac{(m-1)a}{x} = \frac{(m-1)b}{y} = -mc \text{ ....} (32);$$
and eliminating a, b, and c, the parameters from the above four equations, we find
$$x^2 + y^2 = \frac{1-m}{m} \text{ ....} (33),$$
which agrees with (27), as it ought to do. Thus far the Rule.

But if from (30) and the two equations (32) we eliminate x and y, we get
$$(m-1)(a^2+b^2) + mc^2 = 0,$$
which is a reproduction of (31), and serves at once to verify the process which has been pursued, although less directly than does the equation (33), and to illustrate the remark which has been offered.

Lastly, if to the condition (31), connecting the parameters a, b, c, we add any other arbitrary condition, such, for example, as
$$a^2 + b^2 = 1,$$
the straight line defined by the equation $ax + by - c = 0$ will still have for its envelope, or for a part of it, the same equation $x^2 + y^2 = \frac{1-m}{m}$, according to the principle stated in the Rule.

Thus, in the case just mentioned, we have the equations

$$ax + by - c = 0,$$
$$(m-1)(a^2+b^2) + mc^2 = 0 \text{ .... } (34),$$
$$a^2 + b^2 = 1\ ;$$

and eliminating da, db, dc from the differentials, we get

$$ay = bx,$$

from which four equations, on eliminating a, b, c, there results, as before,

$$x^2 + y^2 = \frac{1-m}{m}.$$

Suppose that the added condition was

$$pa - qb = 0 \text{ .... } (35).$$

Omitting the steps of a very tedious process of elimination and reduction, I shall simply indicate the result. The final equation of the envelope is

$$\{m(x^2+y^2)+m-1\}$$
$$\{(my^2+m-1)p^2 + 2mxypq + (mx^2+m-1)q^2\} = 0 \ldots (36),$$

the first factor of which, equated to 0, gives

$$x^2 + y^2 = \frac{1-m}{m},$$

the equation of a circle as before; the second gives

$$(py+qx)^2 = \frac{1-m}{m}(p^2+q^2);$$

or

$$\frac{py+qx}{\sqrt{(p^2+q^2)}} = \pm\sqrt{\left(\frac{1-m}{m}\right)},$$

and represents two straight lines touching the circle at the two extremities of that diameter which makes with the axes x and y angles whose respective cosines are

$$\frac{q}{\sqrt{(p^2+q^2)}} \text{ and } \frac{p}{\sqrt{(p^2+q^2)}}.$$

What is implied in this interpretation? I think it represents the following circumstance.

If the envelope of a straight line, the parameters in the equation of which are subjected to given conditions, is a complete re-entering curve, the constants representing the cosines of the angles which the straight line in its different

positions makes with the coordinate axes must be susceptible of the whole series of their values from -1 to 1.

But if the relations among the constants are such that some particular values are excluded, a breach of continuity in the curve occurs. Let us suppose that there are two points for which this happens. Then the whole curve, supposed to be a re-entering one, is divided into two portions mutually separated at their extremities by the points in question.

Now the envelope of the tangent to the curve supposed to remain in contact with one of the above portions will be that portion or arc itself, together with those portions of the tangents at the extremities of the arc which may be regarded as continuations of the arc. Hence when there are two arcs of a curve separated at their extremities by intermediate points, the envelope will consist of the two arcs, together with the four branches of the tangent at their four extremities, that is, together with the two complete tangents at the separating points.

Now in the case we have been considering, all the equations of the system are homogeneous with reference to a, b, and c, except the added equation

$$pa - qb = s.$$

Supposing that s does not vanish, this equation may consist with any ratio between a and b, except the ratio of q to p. The straight line represented by the equation $ax + by = c$ may pass, in contact, consistently with the conditions which are imposed upon it, over every part of the circle represented by the equation

$$x^2 + y^2 = \frac{1-m}{m},$$

except the two points in which a and b would have the above ratio, *i.e.* the two points which form the extremities of the diameter which makes with the axis of x an angle whose cosine is $\frac{q}{\sqrt{(p^2+q^2)}}$. Hence the complete envelope consists of the two separated portions of the circle, together with the two complete tangents at the separating points. This is in fact what the general equation (65) implies.

If $s = 0$, all the equations are homogeneous with respect to a, b, and c, and we get, on elimination, one additional equation, viz.

$$\frac{(py + qx)}{\sqrt{(p^2+q^2)}} = \pm\sqrt{\left(\frac{1-m}{m}\right)}.$$

Now this agrees with the equation afforded by the second factor of (36). The envelope is here reducible to the pair of tangents described in the previous section.

2. Let m be less than n, we have then the following theorem.

THEOREM III. If two sets of quantities $x_1, x_2 \ldots x_n$ and $a_1, a_2 \ldots a_m$, m being less than n, are connected by any equations $u_1 = 0$, $u_2 = 0 \ldots u_r = 0$. And if the former set, varying in those equations in subjection to the condition $X = 0$, establish among the other set a relation $A = 0$, together with any $n - m$ additional relations $X_1 = 0$, $X_2 = 0 \ldots X_{n-m} = 0$, among the quantities $x_1, x_2 \ldots x_m$; then the set $a_1, a_2 \ldots a_m$, varying in the original equations $u_1 = 0$, $u_2 = 0 \ldots u_r = 0$, in subjection to the condition $A = 0$, will establish among the quantities $x_1, x_2 \ldots x_n$ a relation $W = 0$, of which, and of the previous $n - m$ relations among those quantities, the relation $X = 0$ will be a consequence, the relation $W = 0$ being moreover the essential condition of the existence of the relation $X = 0$.

If, as before, we represent the function $\lambda_1 u_1 + \lambda_2 u_2 \ldots + \lambda_r u_r$ by U, we have the following set of equations $r + n + 1$ in number, viz.

$$u_1 = 0, \quad u_2 = 0, \ldots u_r = 0, \quad X = 0,$$

$$\frac{d(U+X)}{dx_1} = 0, \quad \frac{d(U+X)}{dx_2} = 0, \ldots \frac{d(U+X)}{dx_n} = 0 \ldots (37);$$

from which, if in the first instance we eliminate $a_1, a_2 \ldots a_m$, $\lambda_1, \lambda_2 \ldots \lambda_r$, we obtain $n + 1 - m$ equations among $x_1, x_2 \ldots x_n$, that is, $n - m$ equations additional to the equation $X = 0$. We shall represent these equations by $X_1 = 0$, $X_2 = 0 \ldots X_{n-m} = 0$. Eliminating $\lambda_1, \lambda_2 \ldots \lambda_r$, $x_1, x_2 \ldots x_n$, we also get the condition $A = 0$.

Now if in the original system $u_1 = 0$, $u_2 = 0, \ldots u_r = 0$, we cause $a_1, a_2 \ldots a_m$ to vary in subjection to the condition $A = 0$, the resulting relation $W = 0$ will, by Theorem II., be such that if $x_1, x_2 \ldots x_n$ vary in the same system, subject to the single condition $W = 0$, or to that condition associated with any other $n - m$ arbitrary conditions among $x_1, x_2 \ldots x_n$, the relation $A = 0$ will result. Moreover this relation $W = 0$ is an essential condition of the existence of $A = 0$.

Hence the relation $W = 0$ is implicitly involved in the system (37); since of that system A is a consequence. But the relations among $x_1, x_2 \ldots x_n$, involved in that system,

are $X=0$, $X_1=0\ldots X_{n-m}=0$. Of these relations, therefore, W is a consequence. Conversely, then, the relation $X=0$ is a necessary consequence of the equations

$$W=0, \quad X_1=0, \quad X_2=0, \ldots X_{n-m}=0 \ldots\ldots(38),$$

that is, of the $n-m$ equations furnished in the process for determining A, and the one essential condition $W=0$, subsequently furnished by A.

Lastly, as the relation $W=0$ is essentially involved in the system (37), it is an essential condition of the existence of $X=0$.

When the relations $X_1=0$, $X_2=0, \ldots X_{n-m}=0$, are identical with the relation $X=0$, then that relation unfettered by any other condition will be established by the variation of $a_1, a_2 \ldots a_m$ in the original system in subjection to $A=0$. W and X are then identical.*

The geometrical application of the above results is sufficiently obvious. The problem to be solved, and the rule furnished by the general theorem, may be thus stated.

The equation or equations of a given locus involving both variable parameters and coordinates, the number of the former being less than the number of the latter; required the conditions among the parameters under which the locus shall, by its successive mutual intersections, either generate a given fixed surface as its envelope, or execute the greatest possible amount of motion in contact with the given fixed surface. Required also, in the latter case, the equation of the envelope actually generated, and the equations of the track marked out by it upon the given fixed surface.

Rule. Determine the equation of the envelope of the moveable locus when the coordinates are regarded as parameters, and *vice versâ*, and the equation of the given fixed surface is regarded as expressing the condition among the parameters thus assumed. Find also by elimination the additional relations among the parameters employed in the above process.

If those relations are identical with the equation of the fixed surface, then the equation of the envelope above determined expresses the condition under which the move-

* I am led to suspect that when there is but a single original equation $u=0$ connecting the two sets of quantities $x_1, x_2, \ldots x_n$ and $a_1, a_2, \ldots a_m$, the equations $X=0$, $W=0$ will together virtually comprise the whole system of relations $X_1=0$, $X_2=0 \ldots X_{n-m}=0$. At present, however, I have not leisure to pursue the inquiry. Should any one else be disposed to take it up, the examination of this point would be productive of interest.

able surface will have the proposed fixed surface for its envelope.

But if the relations are not identical with the equation of the fixed surface, then the equation of the envelope above determined expresses the condition under which the given moveable surface will execute the greatest possible amount of motion in contact with the given fixed surface. And the equation of the envelope actually described may be determined in the ordinary way. The locus of contact will be determined by the equation of the fixed surface, together with the relations above referred to. If, however, there exist but one such relation, the equation of the envelope actually described may be used in its place.

It is probable that the above theory may appear complex from the great number of circumstances involved in its complete exposition. But it belongs to a harmonious and connected system, and one or two examples will suffice to elucidate every difficulty.

Ex. 1. To ascertain if possible the conditions under which the plane, whose equation is

$$ax + by + z = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (39),$$

shall generate the surface whose equation is

$$4xy = 27m(1-z)^2 \ldots\ldots\ldots\ldots\ldots\ldots (40),$$

a and b being the variable parameters.

Here we are directed first to investigate the envelope of (39), subject to the condition (40), when x, y, and z are regarded as parameters, and a and b as coordinates.

Differentiating both equations with reference to x, y, and z, we have

$$adx + bdy + dz = 0,$$

$$4ydx + 4xdy - 54m(z-1)\,dz = 0.$$

Whence, in the usual way,

$$\frac{2y}{a} = \frac{2x}{b} = -27m(z-1) \ldots\ldots\ldots\ldots\ldots (41),$$

from which

$$x = \frac{-27bm(z-1)}{2}, \quad y = \frac{-27am(z-1)}{2}.$$

Substituting these values in (39) and dividing the result by the common factor $y-1$, we have

$$27mab = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (42).$$

Now to ascertain whether under this condition the plane (39)

will generate (40) as its envelope, let us, according to the rule, eliminate a and b from (39) and (41); we get

$$\frac{-2xy}{27m(z-1)}+\frac{-2xy}{27m(z-1)}+z=1,$$

which gives $$4xy = 27m(z-1)^2.$$

This result is identical with (40), whence we infer that (40) will be the true envelope of (39) under the condition determined.

To verify this conclusion, let us take the equations

$$ax+by+z=1 \text{ .... } (43),$$

$$27mab=1 \text{ .... } (44),$$

and seek the envelope of the former when a and b vary in subjection to the latter.

Differentiating with respect to a and b, we have

$$xda+ydb=0,$$

$$27mbda+27madb=0;$$

and, eliminating da and db,

$$ax-by=0 \text{ .... } (45).$$

Now if from this equation and (43) and (44) we eliminate a and b, we get $$27m(1-z)^2=4xy,$$

which agrees with (40).

Ex. 2. To ascertain the conditions under which the sphere whose equation is

$$(x-a)^2+(y-b)^2+z^2=1 \text{ .... } (46)$$

shall if possible have for its envelope the cone whose equation is $$\sqrt{(x^2+y^2)}+z=r \text{ .... } (47),$$

a and b being the variable parameters.

Proceeding according to the Rule, we have

$$(x-a)\,dx+(y-b)\,dy+zdz=0,$$

$$\frac{x}{\sqrt{(x^2+y^2)}}dx+\frac{y}{\sqrt{(x^2+y^2)}}dy+dz=0.$$

Whence, by proceeding as before, we get the two equations

$$\frac{x}{(x-a)\sqrt{(x^2+y^2)}}=\frac{y}{(y-b)\sqrt{(x^2+y^2)}}=\frac{1}{z} \text{ ......} (48).$$

The first of these equations gives

$$\frac{x}{x-a} = \frac{y}{y-b}, \text{ or } bx = ay \text{ .... (49).}$$

We also readily deduce from (48)

$$z = \sqrt{\{(x-a)^2 + (y-b)^2\}},$$

whence, by (46), $z = \sqrt{(1-z)^2}$;

therefore $z = \pm \sqrt{\tfrac{1}{2}}$.

(47) then gives $x^2 + y^2 = (r \pm \sqrt{\tfrac{1}{2}})^2$;

whence, by (49), we find

$$x = \frac{a}{\sqrt{(a^2+b^2)}}(r \pm \sqrt{\tfrac{1}{2}})^2, \quad y = \frac{b}{\sqrt{(a^2+b^2)}}(r \pm \sqrt{\tfrac{1}{2}})^2.$$

And if the values of x, y, and z, thus found, be substituted in (46), we find, after reduction,

$$a^2 + b^2 = (r \pm \sqrt{2})^2 \text{ .... (50)}$$

for the condition between the parameters a and b.

Now we have seen in the course of this investigation that the elimination of a and b between (46) and the two equations of (48), gave us

$$z = \pm \sqrt{\tfrac{1}{2}} \text{ .... (51),}$$

an equation which is not identical with (47). Hence it is not possible that the sphere (46) should generate the cone (47) as its envelope, a and b being the only variable parameters. Equation (50), however, expresses the condition under which the sphere can move in contact with the cone, the equations of the locus of contact being (47) and (51).

To find the equation of the actual envelope, we have

$$(x-a)^2 + (y-b)^2 = 1 - z^2 \text{ .... (52),}$$

$$a^2 + b^2 = (r \pm \sqrt{2})^2 \text{ .... (53).}$$

Differentiating with respect to a and b,

$$(x-a)\,da + (y-b)\,db = 0,$$

$$ada + bdb = 0;$$

whence, eliminating da and db, we get

$$bx = ay.$$

Hence, and from (53), we have

$$a = \frac{x(r \pm \sqrt{2})}{\sqrt{(x^2+y^2)}}, \quad b = y\frac{(r \pm \sqrt{2})}{\sqrt{(x^2+y^2)}};$$

and substituting these values in (52), and reducing, we get

$$\sqrt{(x^2 + y^2)} - \sqrt{(1 - z^2)} = r \pm \sqrt{2}$$

as the equation of the true envelope, which is in fact a pair of tubular rings.

We have seen that the equations of the locus of contact are

$$\sqrt{(x^2 + y^2)} + z = r,$$
$$z = \pm \sqrt{\tfrac{1}{2}}.$$

Now it is easy to shew that these equations represent two circles traced on the above rings, the one at a uniform distance of $\sqrt{\frac{1}{2}}$ above the plane of xy on the inner ring, the other at a like distance below the plane of xy on the outer ring. It must be observed that $\sqrt{(1 - z^2)}$ has the same sign as z.

All these conclusions may be verified by geometrical considerations. It is obvious that a sphere of invariable radius (46), the centre of which is restricted to motion in a plane (since a and b are the only variable parameters), cannot have a cone for its envelope. But it may move so as to preserve contact with the cone; and it is evident that the locus of contact will be a pair of circles, the one above and the other below the plane of xy, and both of them parallel to that plane. It is evident also that the envelope of the sphere, while executing these motions, will consist of two hollow rings, the one girding the cone, the other touching its inner surface.

We may now compare together the different cases which present themselves in the theory which has passed under discussion.

We may state it to be the general object of this investigation, so far as it may be expressed in the language of geometry, to cause a given moveable locus to execute the greatest possible amount of motion in contact with another fixed locus, by the variation of certain constant elements in its equation called parameters.

This is the most general object of the inquiry as respects geometry. It includes the determination of the condition under which a given envelope will be generated, for this will be effected whenever the given moveable locus is permitted to pass in contact over every part of the fixed locus which thus becomes its envelope. It includes also those cases in which the given moveable locus can only have contact with the fixed locus along some unknown line or at some unknown point or points to be determined.

And it appears that in all these cases the required condition among the parameters will be found by seeking the equation of the envelope of the moveable surface, regarding in the equation of the latter the parameters as coordinates and the coordinates as parameters.

When the number of the original parameters is equal to the number of the coordinates, the conditions above determined suffice to cause the moveable locus to generate the fixed locus as its envelope. When the former number exceeds the latter number by m, we may introduce at liberty m arbitrary equations among the parameters. When it falls short of it by m, we obtain in the process m additional, but not necessarily new, equations among the coordinates which define the trace of the moveable surface upon the fixed one.

Such is the theory of the inverse problem of envelopes. From the direct one it differs in some important particulars. In the direct problem the relation of the number of parameters to that of the coordinates is comparatively unimportant; the number of relations to which the parameters are subject is equally unimportant. One method (Lagrange's one of indeterminate multipliers) serves for all cases. In the inverse method we have necessarily but two things given, the equation of the moveable and the equation of the fixed surface. The relation of the number of the parameters to that of the coordinates is here however all-important. It presents us three cases for consideration, the characters of which are quite distinct. The arbitrary functions which in one of those cases enter into the solution, seem to indicate an approximation between the results of the differential and those of the integral calculus.

Application of the Method to some Problems connected with the Wave Surface.

It is known that the wave surface is the envelope of the plane whose equation is

$$lx + my + nz = v \ldots\ldots\ldots\ldots (54),$$

the parameters l, m, n, v being made to vary in subjection to the conditions

$$l^2 + m^2 + n^2 = 1 \ldots\ldots\ldots\ldots (55),$$

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0 \ldots\ldots\ldots (56).$$

Its equation is

$$\frac{x^2}{r^2-a^2}+\frac{y^2}{r^2-b^2}+\frac{z^2}{r^2-c^2}=1 \quad\ldots\ldots\ldots\ldots (57),$$

in which r^2 stands for $x^2+y^2+z^2$.

There are in the above case three coordinates x, y, z, and four parameters l, m, n, v. It appears, then, from the theory developed in the previous pages, that if we seek the conditions under which the wave surface (57) shall be generated by the ultimate intersections of the plane (54), we shall obtain one *essential* condition among the parameters to which we shall be permitted to add one other condition chosen *arbitrarily*. Let us consider these two equations as a system involving one arbitrary element. Then, as this system represents the most general solution of the problem proposed, it is obvious that the two equations (55) and (56), expressing the conditions by which the wave surface has *actually* been determined, must be a particular case of the above system obtained by giving a particular form to the arbitrary equation which it involves. The solution of the inverse problem will therefore be something more than a reproduction of the original equations of condition. It will shew us what element in those conditions is the essential one, and what element is arbitrary and might be rejected or replaced.

We are then to seek the conditions under which the plane

$$lx+my+nz=v \quad\ldots\ldots\ldots\ldots\ldots\ldots (58)$$

shall have for its envelope the surface whose equation is

$$\frac{x^2}{r^2-a^2}+\frac{y^2}{r^2-b^2}+\frac{z^2}{r^2-c^2}=1 \quad\ldots\ldots\ldots\ldots (59),$$

l, m, n, v, being parameters.

Regarding x, y, z, r as parameters, and adding the equation

$$x^2+y^2+z^2=r^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (60),$$

we obtain, on differentiating with respect to x, y, z, r,

$$ldx+mdy+ndz=0,$$

$$\frac{xdx}{r^2-a^2}+\frac{ydy}{r^2-b^2}+\frac{zdz}{r^2-c^2}=r\left\{\frac{x^2}{(r^2-a^2)^2}+\frac{y^2}{(r^2-b^2)^2}+\frac{z^2}{(r^2-c^2)^2}\right\}dr,$$

$$xdx+ydy+zdz=rdr.$$

Multiply the first equation by λ, the third by μ, and from the result subtract the second. Then equating to 0 the

coefficients of dx, dy, dz, dr, we have

$$\lambda l + \mu x = \frac{x}{r^2 - a^2} \text{ .... } (61),$$

$$\lambda m + \mu y = \frac{y}{r^2 - b^2} \text{ .... } (62),$$

$$\lambda n + \mu z = \frac{z}{r^2 - c^2} \text{ .... } (63),$$

$$\mu = \frac{x^2}{(r^2 - a^2)^2} + \frac{y^2}{(r^2 - b^2)^2} + \frac{z^2}{(r^2 - c^2)^2} \text{ .... } (64).$$

$$(61) \times \frac{x}{r^2 - a^2} + (62) \times \frac{y}{r^2 - b^2} + (63) \times \frac{z}{r^2 - c^2}$$

gives, by (59) and (64),

$$\lambda\left(\frac{lx}{r^2 - a^2} + \frac{my}{r^2 - b^2} + \frac{nz}{r^2 - c^2}\right) + \mu = \mu,$$

therefore $$\frac{lx}{r^2 - a^2} + \frac{my}{r^2 - b^2} + \frac{nz}{r^2 - c^2} = 0 \text{ .... } (65).$$

Again, $(61) \times l + (62) \times m + (63) \times n$ gives, on putting ρ^2 for $l^2 + m^2 + n^2$,

$$\lambda\rho^2 + \mu v = 0 \text{ .... } (66).$$

Lastly, $(61) \times x + (62) \times y + (63) \times z$ gives

$$\lambda v + \mu r^2 = 1 \text{ .... } (67).$$

From the two last equations, we find

$$\lambda = \frac{-v}{r^2\rho^2 - v^2}, \quad \mu = \frac{\rho^2}{r^2\rho^2 - v^2}.$$

Substitute these values in (61), and we get

$$\frac{\rho^2 x - lv}{r^2\rho^2 - v^2} = \frac{x}{r^2 - a^2};$$

whence, if we determine the value of x, we shall find

$$\frac{x}{r^2 - a^2} = \frac{lv}{v^2 - a^2\rho^2} \text{ .... } (68);$$

and, in like manner, from (62) and (63) we shall have

$$\frac{y}{r^2 - b^2} = \frac{mv}{v^2 - b^2\rho^2},$$

$$\frac{z}{r^2 - c^2} = \frac{nv}{v^2 - c^2\rho^2}.$$

Multiply (68) and the two following equations by l, m, n, respectively, and attending to (65), we shall have, on dividing the result by v,

$$\frac{l^2}{v^2-a^2\rho^2}+\frac{m^2}{v^2-b^2\rho^2}+\frac{n^2}{v^2-c^2\rho^2}=0 \text{ ....(69)},$$

wherein ρ^2 stands for $l^2+m^2+n^2$.

Such is the necessary equation of condition among the parameters l, m, n, v, in order that the plane 91 may have the wave surface 94 for its envelope. With this equation of condition we can, as has been observed, associate any other arbitrary equation. If we choose for this purpose the equation $l^2+m^2+n^2=1$, the equation (69) is then reducible to

$$\frac{l^2}{v^2-a^2}+\frac{m^2}{v^2-b^2}+\frac{n^2}{v^2-c^2}=0,$$

and the original conditions (55) and (56) are produced.

It may be interesting, however, to verify the equation (69) and to prove its single sufficiency, by directly investigating the envelope of the plane given, subject to the condition which that equation expresses.

Our equations are

$$lx+my+nz=v \text{ ....(70)},$$

$$\frac{l^2}{v^2-a^2\rho^2}+\frac{m^2}{v^2-b^2\rho^2}+\frac{n^2}{v^2-c^2\rho^2}=0 \text{ ....(71)},$$

in which $\rho^2=l^2+m^2+n^2$. These equations we may differentiate with reference to l, m, n, v, and apply, as before, the method of indeterminate multipliers. The result, as I have found by actually carrying out this process, is the equation of the wave surface (57). But that result is obtained with a little more convenience as follows.

Let $\frac{l}{\rho}=l'$, $\frac{m}{\rho}=m'$, $\frac{n}{\rho}=n'$, $\frac{v}{\rho}=v'$. Then the equations (70), (71) may be replaced by the following system:

$$l'x+m'y+n'z=v' \text{ ....(72)},$$

$$\frac{l'^2}{v'^2-a^2}+\frac{m'^2}{v'^2-b^2}+\frac{n'^2}{v'^2-c^2}=0 \text{ ....(73)},$$

$$l'^2+m'^2+n'^2=1 \text{ .... (74)}.$$

Now these are the very equations from which the equation of the wave surface is known to be deduced, excepting that l', m', n', v', stand for l, m, n, v, a difference which does not affect the result.

As to the essential condition (69), we are permitted to annex one other condition arbitrary in its character; we may assume for this condition that one of the parameters l, m, n, v is constant. It would hence appear that in the equation (69) it is not necessary to suppose more than three of the parameters to vary, and whichsoever three of the set we choose, the wave surface will be the resulting envelope of the plane $lx + my + nz = v$. This conclusion may be established by other considerations. It might in fact be deduced as a consequence of the homogeneity of the two equations (70) and (71) with respect to the four parameters l, m, n, v. But this circumstance is accidental. The variation of any three of the parameters in (71), or of the whole four, subject or not to an additional relation, would secure the object proposed quite independently of any condition of homogeneity in the equation given.

In (70) and (71) for $\frac{l}{v}, \frac{m}{v}, \frac{n}{v}$, put λ, μ, ν, respectively, and for $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ write α, β, γ.

Then it will appear that the envelope of the plane

$$\lambda x + \mu y + \nu z = 1 \text{ .... } (75),$$

subject to the condition

$$\frac{\alpha^2\lambda^2}{\rho^2 - \alpha^2} + \frac{\beta^2\mu^2}{\rho^2 - \beta^2} + \frac{\gamma^2\nu^2}{\rho^2 - \gamma^2} = 0 \text{ .... } (76),$$

in which $\rho^2 = \lambda^2 + \mu^2 + \nu^2$ will be the wave surface, whose equation is

$$\frac{x^2}{r^2 - \dfrac{1}{\alpha^2}} + \frac{y^2}{r^2 - \dfrac{1}{\beta^2}} + \frac{z^2}{r^2 - \dfrac{1}{\gamma^2}} = 1 \text{ .... } (77).$$

Now (71) may be reduced to the form

$$\frac{\lambda^2}{\rho^2 - \alpha^2} + \frac{\mu^2}{\rho^2 - \beta^2} + \frac{\nu^2}{\rho^2 - \gamma^2} = 1 \text{ .... } (78),$$

which would be the equation of the wave surface if λ, μ, ν were regarded as parameters. On comparing (77) and (78) it becomes apparent that the polar reciprocal of a wave surface is another wave surface whose axes are the reciprocals of the axes of the given one. The equation (75) exhibits the relation connecting the coordinates λ, μ, ν of the one surface with the coordinates x, y, z of the other,

and indicates that the surface of transformation is a sphere whose radius is unity, and whose centre is placed at the common origin of coordinates. The same conclusion has, I believe, been deduced by MacCullagh from the known reciprocal properties of the ellipsoid.

If two only of the parameters l, m, n, v were permitted to vary, it would no longer be possible that the plane (70) should generate the wave surface as the locus of its successive intersections, but we might still ascertain the condition under which the plane might move in contact with the surface and the equations of that line of double curvature, which would be the locus of its successive points of contact.

As another example, let us investigate the conditions under which the wave surface can be generated by the mutual intersections of ellipsoids, concentric with itself, and having axes coincident with the axes of coordinates.

We will suppose the equation of the ellipsoids to be

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1 \text{ .... } (79),$$

α, β, γ, the semiaxes, being the variable parameters.

We must in this equation, and in the equation of the wave surface, now regard x, y, z as the variable parameters. But as x, y, z only enter these equations under the forms x^2, y^2, z^2, it will be convenient to substitute x for x^2, y for y^2, z for z^2, and r for r^2, previous to differentiation and elimination. This merely amounts to regarding x^2, y^2, z^2 as the variable parameters instead of x, y, z. We shall also, for simplicity, make

$$\frac{1}{\alpha^2} = l, \ \frac{1}{\beta^2} = m, \ \frac{1}{\gamma^2} = n,$$

and we shall write a for a^2, b for b^2, c for c^2.

Our equations will then be

$$lx + my + nz = 1 \text{ .... } (80),$$

$$x + y + z = r \text{ .... } (81),$$

$$\frac{x}{r-a} + \frac{y}{r-b} + \frac{z}{r-c} = 1 \text{ .... } (82).$$

If now we differentiate these equations with reference to x, y, z, r, we shall in the ordinary way deduce

$$\lambda l + \mu = \frac{1}{r - a}, \quad \lambda m + \mu = \frac{1}{r - b}, \quad \lambda n + \mu = \frac{1}{r - c} \ldots (83),$$

$$\mu = \frac{x}{(r - a)^2} + \frac{y}{(r - b)^2} + \frac{z}{(r - c)^2} \ldots\ldots\ldots (84).$$

Multiply the three equations of (83) by $\frac{x}{r-a}$, $\frac{y}{r-b}$, and $\frac{z}{r-c}$ respectively, and reducing by (82) and (84), we get

$$\lambda\left(\frac{lx}{r - a} + \frac{my}{r - b} + \frac{nz}{r - c}\right) + \mu = \mu;$$

therefore $$\frac{lx}{r - a} + \frac{my}{r - b} + \frac{nz}{r - c} = 0 \ldots\ldots\ldots\ldots\ldots (85).$$

From equation (82) subtract (81) divided by r, the result is

$$\frac{ax}{r - a} + \frac{by}{r - b} + \frac{cz}{r - c} = 0 \ldots\ldots\ldots\ldots\ldots (86).$$

From the two last equations and (82), regarded as linear with respect to $\frac{x}{r-a}$, $\frac{y}{r-b}$, $\frac{z}{r-c}$, we get

$$\frac{x}{r - a} = \frac{p}{p + q + s}, \quad \frac{y}{r - b} = \frac{q}{p + q + s}, \quad \frac{z}{r - b} = \frac{s}{p + q + s} \ldots (87),$$

in which, for brevity,

$$p = cm - bn, \quad q = an - cl, \quad s = bl - am.$$

From (87) we have

$$lx + my + nz = \frac{lp(r - a) + mq(r - b) + ns(r - c)}{p + q + s},$$

$$= \frac{r(lp + mq + ns) - lap - mbq - ncs}{p + q + s}.$$

Substituting for p, q, and s their values, and observing that $lx + my + nz = 1$, we get

$$1 = \frac{-la(cm - bn) - mb(an - cl) - nc(bl - am)}{cm - bn + an - cl + bl - am},$$

or, as it may be written,

$$(la + 1)(cm - bn) + (mb + 1)(an - cl) + (nc + 1)(bl - am) = 0,$$

wherein, if for l, m, n we substitute their values and replace

a by a^2, b by b^2, c by c^2, we get

$$\left(\frac{a^2}{\alpha^2}+1\right)\left(\frac{c^2}{\beta^2}-\frac{b^2}{\gamma^2}\right)+\left(\frac{b^2}{\beta^2}+1\right)\left(\frac{a^2}{\gamma^2}-\frac{c^2}{\alpha^2}\right)+\left(\frac{c^2}{\gamma^2}+1\right)\left(\frac{b^2}{\alpha^2}-\frac{a^2}{\beta^2}\right)=0$$

as the expression of the required condition among the semi-axes α, β, γ.

I found it somewhat difficult to verify this result, but at length succeeded in doing so by a process of which I shall indicate the principal steps.

If in the equation of the ellipsoid and the equation expressing the condition among the axes we make the same substitutions as before, we get

$$lx + my + nz = 1 \text{ .... } (89),$$

$$(la+1)(cm-bn)+(mb+1)(an-cl)+(nc+1)(bl-am)=0 \text{...} (90),$$

in which l, m, n are to be regarded as variable parameters. We hence deduce

$$\lambda x + a(cm-bn) + b(nc+1) - c(mb+1) = 0 \text{ ... } (91),$$

and two other similar equations.

Again, from (90) and the identical equation

$$a(cm-bn)+b(an-cl)+c(bl-am)=0,$$

we get

$$\frac{cm-bn}{c(mb+1)-b(nc+1)}=\frac{an-cl}{a(nc+1)-c(la+1)}=\frac{bl-am}{b(la+1)-a(mb+1)}.$$

Let any of the above fractions be represented by $\frac{1}{k}$. Then (91) may be put under the form

$$\frac{\lambda x}{cm-bn}+a=k \text{.....................} (92),$$

or

$$\lambda x + a(cm-bn) = k(cm-bn) \text{ .... } (93).$$

In like manner, we have

$$\lambda y + b(an-cl) = k(an-cl),$$

$$\lambda z + c(bl-am) = k(bl-am).$$

Adding these equations, we have

$$\lambda r = k\{(cm-bn)+an-cl+bl-am\} \text{ ......} (94).$$

Again, multiplying the same equations by l, m, n, respectively, and adding, we get by (89),

$$\lambda + la(cm-bn) + mb(an-cl) + nc(bl-am) = 0,$$

whence, by (90),

$$\lambda = cm - bn + an - cl + bl - am \text{ .... } (95),$$

and substituting in (94), we get

$$r = k.$$

Substitute this value of k in (92), and we get

$$\frac{\lambda x}{cm - bn} = r - a,$$

therefore $$\frac{x}{r - a} = \frac{cm - bn}{\lambda}.$$

Similarly $$\frac{y}{r - b} = \frac{an - cl}{\lambda},$$

$$\frac{z}{r - c} = \frac{bl - am}{\lambda};$$

and adding these results together, and substituting for λ its value given in (95), we have

$$\frac{x}{r - a} + \frac{y}{r - b} + \frac{z}{r - c} = 1.$$

Replacing then x by x^2, y by y^2, &c., we have

$$\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2} = 1,$$

the equation desired.

Application to Inverse Problems of Maxima and Minima.

That problems of maxima and minima and those of envelopes are intimately connected, must have occurred to every one who has particularly attended to them. But the precise nature of their connexion does not appear to have been pointed out.

Let us confine ourselves to the supposition that there are but two variables x and y, and let us consider by what process we should seek the maximum value of a proposed function $\phi(x, y)$ or u, the variables being subject to the condition $\psi(x, y) = 0$.

That process consists in eliminating the variables x, y, and their differentials dx, dy, from the equations

$$u = \phi(x, y) \text{ .... } (96),$$

$$\psi(x, y) = 0 \text{ .... } (97),$$

$$\frac{d\phi(x,y)}{dx}dx + \frac{d\phi(x,y)}{dy}dy = 0,$$

$$\frac{d\psi(x,y)}{dx}dx + \frac{d\psi(x,y)}{dy}dy = 0.$$

The result will be a relation connecting u with the constants which enter into $\phi(x, y)$ and $\psi(x, y)$. We will however consider it solely with reference to u and the constants which enter into $\phi(x, y)$.

Now the process above explained is precisely that to which we should be led if, regarding x and y as parameters, and the abovementioned constants as coordinates, we sought the envelope of (96) subject to the condition (97). The converse problem of maxima and minima therefore, or the discovery of the condition (97) when the form of the function $\phi(x, y)$ to be made a maximum, and also its maximum value or the relation which connects u with the constants in $\phi(x, y)$ are known, will be solved in the same way as the inverse problem of envelopes.

The reasoning here exemplified is applicable, *mutatis mutandis*, to cases in which the number of variables exceeds two. We are thus conducted to the following rule.

To ascertain the condition which must connect the variables $x_1, x_2 \ldots x_n$, in order that a proposed function $\phi(x_1, x_2 \ldots x_n, a_1, a_2 \ldots a_m)$ may have its maximum or minimum value determined by a proposed relation $\chi(u, a_1, a_2 \ldots a_m) = 0$, connecting that maximum or minimum value u with the constants $a_1, a_2 \ldots a_m$, which enter into the proposed function.

Rule. Determine in the ordinary way the conditions under which the function

$$u - \phi(x_1. x_2 \ldots x_n, a_1, a_2 \ldots a_m)$$

shall obtain its maximum or minimum value, regarding $u, a_1, a_2 \ldots a_m$ as variables subject to the conditions

$$\chi(u, a_1, a_2 \ldots a_m) = 0.$$

The result will be a relation between $x_1, x_2 \ldots x_n$, which we will represent by $\psi(x_1, x_2 \ldots x_n) = 0$.

If $m + 1 = n$, the above is the relation sought.

If $m + 1 < n$, the above relation still suffices, but we can also add to it any $n - m - 1$ arbitrary relations among $x_1, x_2 \ldots x_n$.

If $m + 1 > n$, we cannot in general effect our object.

It may suffice to give a single example of this theory. We choose a statical problem.

One end of a straight beam rests against a vertical wall, the other rests upon an unknown curve. Required the form of that curve in order that, in consistence with the laws of mechanics, a given relation may exist between the length of the beam and the height of its centre of gravity.

Let the axis of y coincide with the vertical line in which the top of the beam is found, and let the altitude of the top be y'; also let x and y be the coordinates of its other extremity which moves on the unknown curve. Also let l be the length of the beam, and $\frac{1}{2}u$ the height of its centre of gravity. Then we have

$$l^2 = x^2 + (y' - y)^2,$$

and, by the principles of statics,

$$u = y + y' = \text{a minimum}.$$

If from these equations we eliminate y', we get

$$x^2 + (2y - u)^2 = l^2.$$

This is an equation connecting the quantity u, which is to be made a minimum with the variable coordinates xy and the length l. If we knew the relation between x and y, we should in the ordinary way be able to determine the relation between the minimum value of u and l. This would be the direct problem. But in the present instance the relation between u and l is supposed to be given, and that between x and y to be sought.

Let the relation given between u and l be represented by $\chi(u, l) = 0$, we have then the two equations

$$x^2 + (2y - u)^2 = l^2,$$
$$\chi(u, l) = 0.$$

And if from these equations and their differentials relative to u and l, we eliminate the latter quantities, we shall obtain the required relation between x and y.

Thus if the relation between u and l be

$$u = al,$$

we get the following equations,

$$x^2 + (2y - u)^2 - l^2 = 0,$$
$$u - al = 0,$$
$$(2y - u)\,du + l\,dl = 0,$$
$$du - a\,dl = 0,$$

the two last of which give

$$l + a(2y - u) = 0.$$

From this equation and the two first of the previous system, we get, by elimination of u and l,

$$y = \pm \frac{\sqrt{(a^2 - 1)}}{2} x.$$

This indicates that the beam rests upon a straight line.

Suppose in the next place that the relation between u and l is of the form

$$8pu - p^2 - 16l^2 = 0,$$

a form to which the solution of a certain direct problem leads. We get the system

$$x^2 + (2y - u)^2 - l^2 = 0,$$
$$8pu - p^2 - 16l^2 = 0,$$
$$p - 4(u - 2y) = 0,$$

the last equation obtained by eliminating the differentials. From these we readily get

$$u = \frac{p + 8y}{4}, \quad l^2 = \frac{8pu - p^2}{16} = \frac{p^2 + 16py}{16};$$

and, substituting in the first equation of the system, we find

$$x^2 + \frac{p^2}{16} = \frac{p^2 + 16py}{16},$$

or

$$x^2 = py,$$

whence the curve on which the beam rests is a parabola with its axis vertical.

It is theoretically quite as easy to ascertain the forms of surfaces on which bodies of a given shape shall rest in equilibrium according to given laws. Probably many of the questions which have been discussed in this paper might also be solved by direct methods. But it has been my special object in the preceding investigations to exhibit those reciprocal relations in the theorems of the Differential Calculus which, at the same time that they lead to practical ends, afford interesting subjects of inquiry and contemplation.

The application of these principles to the Calculus of Variations opens an interesting field of investigation, in which I regret that I have not at present leisure to engage.

P.S.—In the interval between the forwarding of the above paper to this *Journal*, and the appearance of its First Part, (May 1852), the general principle which it involves was independently announced, at least for homogeneous functions, by Mr. Sylvester (*Journal*, Feb. 1852, p. 74). From a statement which has been made to me by that gentleman, I believe that the priority of discovery, as well as of publication, is due to him, and I willingly resign my own claims in favour of one who has done so much to enrich the kindred branches of analysis.

ON CERTAIN THEOREMS IN THE CALCULUS OF OPERATIONS.

By W. Spottiswoode, M.A., Balliol College, Oxford.

In the *Philosophical Transactions* for the year 1844, Professor Boole enunciated certain theorems relating to the operative symbol $D = x\dfrac{d}{dx}$, and applied them to the integration of a large class of differential equations, and to the evaluation of certain definite integrals: and in the *Cambridge and Dublin Mathematical Journal*, Nov. 1851, Mr. Carmichael has extended this method by demonstrating the analogous theorems relating to the operative symbol

$$\nabla = x_1\frac{d}{dx_1} + x_2\frac{d}{dx_2} + \dots$$

I propose to make some still further extensions: to the cases (1) in which the order of the variables by which the symbols of differentiation are multiplied is not the same as that of the variables with respect to which the differentiations are to be performed; (2) in which the variables by which the symbols of differentiation are multiplied are any linear functions of the given variables.

Section I.

Suppose that the series

$$\begin{array}{c} i_1, i_2, \dots \\ j_1, j_2, \dots \\ \dots\dots \end{array}$$

represents any permutations of the series

$$1, 2, \dots$$

and that ∇_i, ∇_j, ... are operative symbols analogous to ∇, thus:

$$\nabla_i = x_{i_1}\frac{d}{dx_1} + x_{i_2}\frac{d}{dx_2} + \dots$$

$$\nabla_j = x_{j_1}\frac{d}{dx_1} + x_{j_2}\frac{d}{dx_2} + \dots$$

$$\dots\dots\dots\dots\dots\dots$$

Again, adopting a similar notation, the permutation

$$j_1, j_2, \dots$$

of the series $\qquad i_1, i_2, \dots$

will be represented by the series

$$j_{i_1}, j_{i_2}, \dots\dots\dots\dots \quad (3),$$

and the corresponding operative symbol by $\nabla_{j,i}$; so that

$$\nabla_{j,i} = x_{j_{i_1}} \frac{d}{dx_1} + x_{j_{i_2}} \frac{d}{dx_2} + \ldots\ldots\ldots\ldots\ldots \quad (4),$$

and generally

$$\nabla_{l,..j,i} = x_{l..j_{i_1}} \frac{d}{dx_1} + x_{l..j_{i_2}} \frac{d}{dx_2} + \ldots\ldots\ldots\ldots \quad (5).$$

And further, writing

$$\left.\begin{array}{l} \nabla_{..j+i} = \ldots\, x_{j_1} x_{i_1} \dfrac{d^n}{dx_1{}^n} + \ldots\, x_{j_2} x_{i_2} \dfrac{d^n}{dx_2{}^n} + \ldots \\ + (\ldots\, x_{j_1} x_{i_2} + \ldots\, x_{j_2} x_{i_1} + \ldots) \dfrac{d^n}{dx_1\, dx_2 \ldots} + \ldots \end{array}\right\} \ldots\ldots (6),$$

it will be found that $\nabla_{j+i} = \nabla_j \nabla_i - \nabla_{j,i}$ ………………… (7).

Similarly,

$$\nabla_{k+j+i} = \nabla_k \nabla_j \nabla_i - \nabla_{k,j} \nabla_i - \nabla_j \nabla_{k,i} - \nabla_k \nabla_{j,i} + \nabla_{k,j,i} + \nabla_{j,k,i} \ldots (8),$$

and so on for any number of operations. But on account of the non-interchangeability of the order of the permutations, it does not seem generally possible to give a concise expression for the operation $\nabla_{l+..j+i}$.

If, however, the permutations are of such a character as to admit of an interchange of order, the operation in question may be expressed in a very brief and elegant way. Suppose, then, that $P(i)$, $P(j)$ represent the permutations (1), and $P(j, i)$, … permutations of the form (3); then, whenever either the second or the first and last conditions of the following system are satisfied,

$$P(j, k) = P(k, j), \quad P(k, i) = P(i, k), \quad P(i, j) = P(j, i) \ldots (9),$$

there will result

$$\nabla_{j,k} = \nabla_{k,j}, \quad \nabla_{k,i} = \nabla_{i,k}, \quad \nabla_{i,j} = \nabla_{j,i} \ldots\ldots (10).$$

And if, besides, either the first condition or the last two conditions of the following system be satisfied,

$$\left.\begin{array}{l} P\{i, (j, k)\} = P\{(j, k), i\} \\ P\{j, (k, i)\} = P\{(k, i), j\} \\ P\{k, (i, j)\} = P\{(i, j), k\} \end{array}\right\} \ldots\ldots\ldots\ldots (11),$$

the equation (8) may be written in the following symmetrical form,

$$\nabla_{k+j+i} = \nabla_k \nabla_j \nabla_i - \nabla_i \nabla_{j,k} - \nabla_j \nabla_{k,i} - \nabla_k \nabla_{i,j} + 2\nabla_{i,j,k} \ldots (12).$$

This may, for convenience, be called the symmetrical case.

It is not difficult to form the permutational equations corresponding to (9) and (11) for the four systems $P(i)$, $P(j)$, $P(k)$, $P(l)$; and when they are satisfied, there will result

$$\left.\begin{array}{c}\nabla_{l+k+j+i} = \nabla_l \nabla_k \nabla_j \nabla_i \\ - \nabla_j \nabla_k \nabla_{i,l} - \nabla_k \nabla_i \nabla_{j,l} - \nabla_i \nabla_j \nabla_{k,l} \\ - \nabla_i \nabla_l \nabla_{j,k} - \Delta_j \nabla_l \nabla_{k,i} - \nabla_k \nabla_l \nabla_{i,j} \\ + 2\nabla_i \nabla_{j,k,l} + 2\nabla_j \nabla_{k,i,l} + 2\nabla_k \nabla_{i,j,l} + 2\nabla_l \nabla_{i,j,k} \\ + \nabla_{i,l} \nabla_{j,k} + \nabla_{j,l} \nabla_{k,i} + \nabla_{k,l} \nabla_{i,j} \\ - 6\nabla_{l,k,j,i}\end{array}\right\} \ldots (13).$$

The analogy between the structure of these functions and that of determinants suggests the following symbolical notation:

$$\left.\begin{array}{c}(i, i) = \nabla_i,\ (j, j) = \nabla_j,\ (k, k) = \nabla_k, \ldots \\ (j, k)(k, j) = \nabla_{j,k},\ (k, i)(i, k) = \nabla_{k,i},\ (i, j)(j, i) = \nabla_{i,j}, \ldots \\ (j, k)(l, j)(l, k) = \nabla_{j,k,l},\ (k, i)(l, k)(l, i) = \nabla_{k,i,l}, \\ (i, j)(l, i)(l, j) = \nabla_{i,j,l}, \ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{array}\right\} \ldots (14),$$

in which it will be observed, that whenever a letter occurs twice in the left-hand members of these equations, it is to be inserted once as a suffix in the right-hand members; a remark which will sufficiently explain the general formation of the new symbols. The equations (7), (12), ... may then be written as follows:

$$\nabla_{j+i} = \begin{vmatrix} (i, i) & (i, j) \\ (j, i) & (j, j) \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots (15),$$

$$\nabla_{k+j+i} = \begin{vmatrix} (i, i) & (i, j) & (i, k) \\ (j, i) & (j, j) & (j, k) \\ (k, i) & (k, j) & (k, k) \end{vmatrix} \ldots\ldots\ldots\ldots (16),$$

and generally

$$\nabla_{l+\cdots j+i} = \begin{vmatrix} (i, i) & (i, j) & \ldots & (i, l) \\ (j, i) & (j, j) & \ldots & (j, l) \\ \cdots & \cdots & \cdots & \cdots \\ (l, i) & (l, j) & \ldots & (l, l) \end{vmatrix} \ldots\ldots\ldots\ldots (17).$$

Or, adopting Mr. Sylvester's notation,

$$\nabla_{j+i} = \begin{Bmatrix} i, j \\ i, j \end{Bmatrix}, \quad \nabla_{k+j+i} = \begin{Bmatrix} i, j, k \\ i, j, k \end{Bmatrix}, \ldots$$

$$\nabla_{l+\cdots j+i} = \begin{Bmatrix} i, j, \ldots l \\ i, j, \ldots l \end{Bmatrix}.$$

These formulæ enable us to find a symbolical solution of a certain class of partial differential equations; in fact, if

$$x_1^{\alpha} x_2^{\beta} \ldots \frac{d^n}{dx_1^n} + x_1^{\alpha_1} x_2^{\beta_1} \ldots \frac{d^n}{dx_2^n} + \ldots\ldots$$

$$+ \Sigma x_1^{\lambda} x_2^{\mu} \ldots \frac{d^n}{dx_1^{\rho} dx_2^{\sigma} \ldots} + \ldots = \nabla_{l+\ldots j+i} \ldots\ldots(18),$$

in which the number of quantities $i, j, \ldots l$, is n, and

$$\alpha + \beta + \ldots = \alpha_1 + \beta_1 + \ldots = \lambda + \mu + \ldots = \rho + \sigma + \ldots = n \ldots(19).$$

Then, supposing the order of the permutations to be interchangeable, *i.e.* supposing the case to be a symmetrical one, the operative function on the left-hand side of (18)

$$= \begin{Bmatrix} i, j, \ldots l \\ i, j, \ldots l \end{Bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20):$$

and if there be formed other similar expressions

$$\begin{Bmatrix} i', j', \ldots l' \\ i', j', \ldots l' \end{Bmatrix} \begin{Bmatrix} i'', j'', \ldots l'' \\ i''', j''', \ldots l''' \end{Bmatrix}, \ldots$$

the differential equation

$$\left.\begin{aligned} &(Ax_1^{\alpha} x_2^{\beta} \ldots + Bx_1^{\alpha'} x_2^{\beta'} \ldots + \ldots) \frac{d^n u}{dx_1^n} \\ &+ (Ax_1^{\alpha_1} x_2^{\beta_1} \ldots + Bx_1^{\alpha_1'} x_2^{\beta_1'} \ldots + \ldots) \frac{d^n u}{dx_2^n} \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ (A\Sigma x_1^{\lambda} x_2^{\mu} \ldots + B\Sigma x_1^{\lambda'} x_2^{\mu'} \ldots + \ldots) \frac{d^n u}{dx_1^{\rho} dx_2^{\sigma} \ldots} \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \Theta \end{aligned}\right\} \ldots(21),$$

(Θ being any function of the variables x_1, x_2, ...) may be expressed thus:

$$A \begin{Bmatrix} i, j, \ldots l \\ i, j, \ldots l \end{Bmatrix} + B \begin{Bmatrix} i', j', \ldots l' \\ i', j', \ldots l' \end{Bmatrix} + \ldots = \Theta \ldots\ldots (22),$$

and its symbolical solution will be

$$u = F(\nabla)\, \Theta + F(\nabla)\, 0 \ldots\ldots\ldots\ldots\ldots(23),$$

where $F(\nabla) = \left[A \begin{Bmatrix} i, j, \ldots l \\ i, j, \ldots l \end{Bmatrix} + B \begin{Bmatrix} i', j', \ldots l' \\ i', j', \ldots l' \end{Bmatrix} + \ldots \right]^{-1} \ldots(24).$

Section II.

Let

$$\left.\begin{aligned}
\Xi_1 &= \xi_{11}\frac{d}{dx_1} + \xi_{12}\frac{d}{dx_2} + \dots \\
\Xi_2 &= \xi_{21}\frac{d}{dx_1} + \xi_{22}\frac{d}{dx_2} + \dots \\
&\dots\dots\dots\dots\dots\dots
\end{aligned}\right\}\dots\dots\dots\dots(25),$$

where

$$\left.\begin{aligned}
\xi_{11} &= \alpha_{11}x_1 + \beta_{11}x_2 + \dots, \quad & \xi_{12} &= \alpha_{12}x_1 + \beta_{12}x_2 + \dots \\
\xi_{21} &= \alpha_{21}x_1 + \beta_{21}x_2 + \dots, \quad & \xi_{22} &= \alpha_{22}x_1 + \beta_{22}x_2 + \dots \\
&\dots\dots\dots\dots & &\dots\dots\dots\dots
\end{aligned}\right\}\dots(26).$$

Then if

$$\Xi_{2\cdot1} = \xi_{21}\xi_{11}\frac{d^2}{dx_1^{\,2}} + \xi_{22}\xi_{12}\frac{d^2}{dx_2^{\,2}} + \dots + (\xi_{22}\xi_{11} + \xi_{21}\xi_{12})\frac{d^2}{dx_1dx_2} + \dots(27),$$

$$\left.\begin{aligned}
\Xi_{2\cdot1} &= \Xi_2\Xi_1 - (\alpha_{11}\xi_{21}+\beta_{11}\xi_{22}+\dots)\frac{d}{dx_1} - (\alpha_{12}\xi_{21}+\beta_{12}\xi_{22}+\dots)\frac{d}{dx_2} + \dots \\
&= \Xi_2\Xi_1 - \{(\alpha_{11}\alpha_{21}+\beta_{11}\alpha_{22}+\dots)x_1 + (\alpha_{11}\beta_{21}+\beta_{11}\beta_{22}+\dots)x_2 + \dots\}\frac{d}{dx_1} \\
&\quad - \{(\alpha_{12}\alpha_{21}+\beta_{12}\alpha_{22}+\dots)x_1 + (\alpha_{12}\beta_{21}+\beta_{12}\beta_{22}+\dots)x_2 + \dots\}\frac{d}{dx_2} \\
&\quad - \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\}\dots\dots(28),$$

$$\left.\begin{aligned}
&\alpha_{11}\alpha_{21}+\beta_{11}\alpha_{22}+\dots = \genfrac{}{}{0pt}{}{21}{11}, \quad \alpha_{11}\beta_{21}+\beta_{11}\beta_{22}+\dots = \genfrac{}{}{0pt}{}{21}{12}, \dots \\
&\alpha_{12}\alpha_{21}+\beta_{12}\alpha_{22}+\dots = \genfrac{}{}{0pt}{}{21}{21}, \quad \alpha_{12}\beta_{21}+\beta_{12}\beta_{22}+\dots = \genfrac{}{}{0pt}{}{21}{22}, \dots \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\}\dots(29),$$

all which may be comprised in the following formula:

$$\left.\begin{aligned}
&\left|\begin{vmatrix} \alpha_{21} & \alpha_{22} \dots \\ \beta_{21} & \beta_{22} \dots \\ \dots & \dots \end{vmatrix}\begin{vmatrix} \alpha_{11} & \beta_{11} \dots \\ \alpha_{12} & \beta_{12} \dots \\ \dots & \dots \end{vmatrix}\right| \\
&\quad = \begin{vmatrix} \genfrac{}{}{0pt}{}{21}{11} & \genfrac{}{}{0pt}{}{21}{12} & \dots \\ \genfrac{}{}{0pt}{}{21}{21} & \genfrac{}{}{0pt}{}{21}{22} & \dots \\ \dots & \dots & \end{vmatrix}
\end{aligned}\right\}\dots\dots\dots\dots\dots(30),$$

which equation is to be understood disjunctively, *i.e.* each constituent of the product of the two determinants on the

left-hand side of the equation, developed as indicated in (29), is to be equated to the corresponding constituent of the determinant on the right-hand side. Then

$$\left.\begin{aligned}\Xi_{2+1}&=\Xi_2\Xi_1-\left\{\left(\genfrac{}{}{0pt}{}{21}{11}\,x_1+\genfrac{}{}{0pt}{}{21}{12}\,x_2+\ldots\right)\frac{d}{dx_1}+\left(\genfrac{}{}{0pt}{}{21}{21}\,x_1+\genfrac{}{}{0pt}{}{21}{22}\,x_2+\ldots\right)\frac{d}{dx_2}+\ldots\right\}\\&=\Xi_2\Xi_1-\Xi_{2,1}\end{aligned}\right\}\ldots\ldots(31),$$

suppose. In the same way we might write

$$\left.\begin{aligned}&\begin{vmatrix}\alpha_{31} & \alpha_{32}\ldots\\ \beta_{31} & \beta_{32}\ldots\\ \ldots & \ldots\end{vmatrix}\;\begin{vmatrix}\alpha_{21} & \alpha_{22}\ldots\\ \beta_{21} & \beta_{22}\ldots\\ \ldots & \ldots\end{vmatrix}\;\begin{vmatrix}\alpha_{11} & \beta_{11}\ldots\\ \alpha_{12} & \beta_{12}\ldots\\ \ldots & \ldots\end{vmatrix}\\ &\qquad=\begin{vmatrix}\genfrac{}{}{0pt}{}{321}{11} & \genfrac{}{}{0pt}{}{321}{12} & \ldots\\ \genfrac{}{}{0pt}{}{321}{21} & \genfrac{}{}{0pt}{}{321}{22} & \ldots\\ \ldots & \ldots & \ldots\end{vmatrix}\end{aligned}\right\}\ldots\ldots\ldots(32),$$

and there would result

$$\Xi_{3+2+1}=\Xi_3\Xi_2\Xi_1-\Xi_{3,2}\Xi_1-\Xi_2\Xi_{3,1}-\Xi_3\Xi_{2,1}+\Xi_{3,2,1}+\Xi_{2,3,1}.\ .(33).$$

And similarly, the process might be carried on to operations of higher orders. To this class of *index symbols* there corresponds *symmetrical cases* analogous to those mentioned in § 1. For supposing that, instead of (28), we consider the following equation,

$$\left.\begin{aligned}\Xi_{1+2}&=\Xi_1\Xi_2-(\alpha_{21}\xi_{11}+\beta_{21}\xi_{12}+\ldots)\frac{d}{dx_1}-(\alpha_{22}\xi_{11}+\beta_{22}\xi_{12}+\ldots)\frac{d}{dx_2}-\ldots\\&=\Xi_1\Xi_2-\{(\alpha_{21}\alpha_{11}+\beta_{21}\alpha_{12}+\ldots)x_1+(\alpha_{21}\beta_{11}+\beta_{21}\beta_{12}+\ldots)x_2+\ldots\}\frac{d}{dx_1}\\&\qquad-\{(\alpha_{22}\alpha_{11}+\beta_{22}\alpha_{12}+\ldots)x_1+(\alpha_{22}\beta_{11}+\beta_{22}\beta_{12}+\ldots)x_2+\ldots\}\frac{d}{dx_2}\\&\qquad-\ldots\ldots\ldots\ldots\\&=\Xi_1\Xi_2-\left\{\left(\genfrac{}{}{0pt}{}{12}{11}\,x_1+\genfrac{}{}{0pt}{}{12}{12}\,x_2+\ldots\right)\frac{d}{dx_1}+\left(\genfrac{}{}{0pt}{}{12}{21}\,x_1+\genfrac{}{}{0pt}{}{12}{22}\,x_2+\ldots\right)\frac{d}{dx_2}+\ldots\right\}\\&=\Xi_1\Xi_2-\Xi_{1,2}\end{aligned}\right\}\ldots\ldots(34),$$

suppose. Then, if the disjunctive equation

$$\left.\begin{matrix}\genfrac{}{}{0pt}{}{12}{11}\;\genfrac{}{}{0pt}{}{12}{12}\ldots=\genfrac{}{}{0pt}{}{21}{11}\;\genfrac{}{}{0pt}{}{21}{12}\ldots\\ \genfrac{}{}{0pt}{}{12}{21}\;\genfrac{}{}{0pt}{}{12}{22}\ldots\quad\genfrac{}{}{0pt}{}{21}{21}\;\genfrac{}{}{0pt}{}{21}{22}\ldots\\ \ldots\ldots\ldots\qquad\ldots\ldots\ldots\end{matrix}\right\}\ldots\ldots\ldots\ldots(35)$$

holds good, there results

$$\Xi_{1,2} = \Xi_{2,1}.$$

And consequently, adopting a notation analogous to that given in (14), *i.e.* writing

$$(1, 1) = \Xi_1, \quad (2, 2) = \Xi_2, \quad (1, 2)(2, 1) = \Xi_{1,2} \ldots (37),$$

we have

$$\Xi_{2+1} = \begin{vmatrix} (1, 1)(1, 2) \\ (2, 1)(2, 2) \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots (38).$$

Similarly, if the disjunctive equation

$$\left.\begin{array}{l} \underset{11}{pq}\ \underset{12}{pq} \ldots = \underset{11}{qp}\ \underset{12}{qp} \ldots \\ \underset{21}{pq}\ \underset{22}{pq} \ldots = \underset{21}{qp}\ \underset{22}{qp} \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (39)$$

hold good for all values of p and q from 1 to 3 inclusive (p and q however always being different); there will result, from the three cases

$$p, q = 2, 3; \quad p, q = 3, 1; \quad p, q = 1, 2 \ldots\ldots (40),$$

the following relations,

$$\Xi_{2,3} = \Xi_{3,2}, \quad \Xi_{3,1} = \Xi_{1,3}, \quad \Xi_{1,2} = \Xi_{2,1} \ldots\ldots\ldots (41).$$

And further, if the system of quantities

$$\left.\begin{array}{l} \underset{11}{pqr}\quad \underset{12}{pqr} \ldots \\ \underset{21}{pqr}\quad \underset{22}{pqr} \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

are independent of the order of the quantities p, q, r, there will result

$$\Xi_{3,2,1} = \Xi_{2,3,1} = \Xi_{1,2,3} \ldots\ldots\ldots\ldots\ldots (43),$$

and consequently (33) becomes

$$\Xi_{3+2+1} = \Xi_3\Xi_2\Xi_1 - \Xi_1\Xi_{2,3} - \Xi_2\Xi_{3,1} - \Xi_3\Xi_{1,2} + 2\Xi_{3,2,1} \ldots (44);$$

and, extending the notation (37), this may be written

$$\Xi_{3+2+1} = \begin{vmatrix} (1, 1)(1, 2)(1, 3) \\ (2, 1)(2, 2)(2, 3) \\ (3, 1)(3, 2)(3, 3) \end{vmatrix} \ldots\ldots\ldots\ldots (45).$$

And generally, if the system

$$\left.\begin{array}{l} \underset{11}{i_1 i_2\ldots i_j} \quad \underset{12}{i_1 i_2\ldots i_j} \ldots \\ \underset{21}{i_1 i_2\ldots i_j} \quad \underset{22}{i_1 i_2\ldots i_j} \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (46),$$

for all values of j from 2 to n inclusive, and for all values of i_1, i_2,...i_j from 1 to m (m being the number of the variables) inclusive, remains unchanged by any permutation of the order of $i_1 i_2\ldots i_j$; the case will be symmetrical, and

$$\Xi_{n+\ldots 2+1} = \left|\begin{array}{l} (1,1)\,(1,2)\ldots(1,n) \\ (2,1)\,(2,2)\ldots(2,n) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ (n,1)\,(n,2)\ldots(n,n) \end{array}\right| \ldots\ldots\ldots (47),$$

which may for convenience be written, as in §. 1, as follows,

$$\left\{\begin{array}{l} 1,\ 2,\ldots n \\ 1,\ 2,\ldots n \end{array}\right\}.$$

This expression will enable us to find a symbolical solution of another class of partial differential equations, namely the following:

$$U\frac{d^n u}{dx_1^{\,n}} + V\frac{d^n u}{dx_2^{\,n}} + \ldots + W\frac{d^n u}{dx_1^{\,\rho}dx_2^{\,\sigma}\ldots} + \ldots = \Theta \ldots (48),$$

where $\rho + \sigma + \ldots = u,$

$$\left.\begin{array}{l} U = AU_1 + BU_2 + \ldots \\ V = AV_1 + BV_2 + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ W = AW_1 + BW_2 + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots\ldots\ldots\ldots (49);$$

the general type of $U_1, U_2,\ldots V_1, V_2,\ldots W_1, W_2,\ldots$ being

$$\left.\begin{array}{l} U = \ (\alpha_1 x_1 + \beta_1 x_2 + \ldots)(\alpha_2 x_1 + \beta_2 x_2 + \ldots)\ldots = \ \xi_1\,\xi_2\ldots\xi_n \\ V = \ (\alpha_1' x_1 + \beta_1' x_2 + \ldots)(\alpha_2' x_1 + \beta_2' x_2 + \ldots)\ldots = \ \xi_1'\xi_2'\ldots\xi_n' \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ W = \Sigma(\alpha_1^{\backprime} x_1 + \beta_1^{\backprime} x_2 + \ldots)(\alpha_2^{\backprime} x_1 + \beta_2^{\backprime} x_2 + \ldots)\ldots = \Sigma\xi_1^{\backprime}\xi_2^{\backprime}\ldots\xi_n^{\backprime} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots (50),$$

and it being further supposed that

$$\xi_1\xi_2\ldots\xi_n\alpha^n + \xi_1'\xi_2'\ldots\xi_n'\beta^n + \ldots + \Sigma\xi_1^{\backprime}\xi_2^{\backprime}\ldots\xi_n^{\backprime}\alpha^\rho\beta^\sigma \ldots (51)$$

is resolvable into linear factors.

Then the equation (41) may be thus transformed:

$$A\begin{Bmatrix}1,\ 2,\dots n\\ 1,\ 2,\dots n\end{Bmatrix}u + B\begin{Bmatrix}1',\ 2',\dots n'\\ 1',\ 2',\dots n'\end{Bmatrix} + \dots = \Theta \dots\dots (52);$$

and, as before, its solution may be thus expressed:

$$u = \Phi(\Xi)\,\Theta + \Phi(\Xi)\,0 \;\dots\dots\dots\dots\dots\; (53),$$

where $\Phi(\Xi) = \left[A\begin{Bmatrix}1,\ 2,\dots n\\ 1,\ 2,\dots n\end{Bmatrix} + B\begin{Bmatrix}1',\ 2',\dots n'\\ 1',\ 2',\dots n'\end{Bmatrix} + \dots\right]^{-1} \dots (54).$

ON A PHYSICAL PROPERTY OF THE GENERATORS OF THE WAVE SURFACE.

By William Walton.

In the May Number of the *Mathematical Journal* for 1852, I have shewn that the wave surface may be generated by the movement of a curve of double curvature, defined by the equations

$$\left.\begin{aligned} \frac{y^2}{\mu} - \frac{z^2}{\nu} &= b^2 - c^2 \\ \frac{z^2}{\nu} - \frac{x^2}{\lambda} &= c^2 - a^2 \\ \frac{x^2}{\lambda} - \frac{y^2}{\mu} &= a^2 - b^2 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots (1),$$

and subjected to pass through the three directors

$$\left.\begin{aligned} x &= 0 \\ y^2 + z^2 &= a^2 \end{aligned}\right\},$$

$$\left.\begin{aligned} y &= 0 \\ z^2 + x^2 &= b^2 \end{aligned}\right\},$$

$$\left.\begin{aligned} z &= 0 \\ x^2 + y^2 &= c^2 \end{aligned}\right\},$$

the condition of passing through these directors being in fact, as there shewn, equivalent to the two algebraical conditions

$$1 + \lambda + \mu + \nu = 0,$$

$$\lambda a^2 + \mu b^2 + \nu c^2 = 0.$$

I propose now to shew that these generators possess a remarkable physical property.

In the paper referred to I have shewn that

$$\left.\begin{aligned} \frac{x^2}{\lambda} &= a^2 - r^2 \\ \frac{y^2}{\mu} &= b^2 - r^2 \\ \frac{z^2}{\nu} &= c^2 - r^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Differentiating the equations (1), we see that

$$dx : dy : dz :: \frac{\lambda}{x} : \frac{\mu}{y} : \frac{\nu}{z},$$

and therefore, by (2),

$$dx : dy : dz :: \frac{x}{a^2 - r^2} : \frac{y}{b^2 - r^2} : \frac{z}{c^2 - r^2} \quad \ldots\ldots (3).$$

But, α, β, γ, being the direction-cosines of the line of vibration at any point x, y, z, of the wave surface, we know that (see Griffin's *Double Refraction*, p. 11)

$$\alpha : \beta : \gamma :: \frac{l}{a^2 - v^2} : \frac{m}{b^2 - v^2} : \frac{n}{c^2 - v^2}.$$

Also (see Griffin's *Double Refraction*, p. 18)

$$v = \frac{x}{l} \cdot \frac{a^2 - v^2}{a^2 - r^2} = \frac{y}{m} \cdot \frac{b^2 - v^2}{b^2 - r^2} = \frac{z}{n} \cdot \frac{c^2 - v^2}{c^2 - r^2}.$$

Hence $$\alpha : \beta : \gamma :: \frac{x}{a^2 - r^2} : \frac{y}{b^2 - r^2} : \frac{z}{c^2 - r^2} \quad \ldots\ldots\ldots (4).$$

From (3) and (4) we have

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{\gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The equations (5) establish the following proposition:

"The line of vibration at any point of the wave-surface is a tangent to the generator which passes through that point."

Or, which amounts to the same thing,

"The tangent at any point whatever of any generator of the wave-surface is a line of vibration at that point of the surface."

Cambridge, July 28, 1852.

ON THE CONDITIONS OF SIMILARITY OF TWO SURFACES OF THE SECOND DEGREE NOT SIMILARLY PLACED.

By Thomas Weddle.

Let $A_1x^2+A_2y^2+A_3z^2+2B_1yz+2B_2xz+2B_3xy=U$......(1),

and $\quad a_1x^2+a_2y^2+a_3z^2+2b_1yz+2b_2xz+2b_3xy=u$......(2),

(where U and u are of the first degree) be the equations to two similar but not similarly placed surfaces of the second degree.

Let (1) be transformed so that the new coordinate axes may be parallel to its principal axes, then its equation will be of the form

$$P_1x^2+P_2y^2+P_3z^2=U' \quad\ldots\ldots\ldots\ldots\ldots(3).$$

In like manner let (2) be referred to parallels to its principal axes, and its equation will become

$$p_1x^2+p_2y^2+p_3z^2=u' \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Now though the two surfaces are no longer referred to the same axes, yet either system of axes may be turned round (carrying its surface along with it), and made to coincide with the other system: if this were done, the two surfaces would then be not only similar but similarly placed, and hence the conditions of similarity are

$$\frac{P_1}{p_1}=\frac{P_2}{p_2}=\frac{P_3}{p_3} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

But (Gregory's *Solid Geometry*, chap. IV.) P_1, P_2, P_3 are the roots of the 'discriminating cubic'

$$(P-A_1)(P-A_2)(P-A_3)-B_1^2(P-A_1)-B_2^2(P-A_2)$$
$$-B_3^2(P-A_3)-2B_1B_2B_3=0\ldots(6),$$

that is, of

$$P^3-(A_1+A_2+A_3)\,P^2+(A_2A_3+A_1A_3+A_1A_2-B_1^2-B_2^2-B_3^2)\,P$$
$$-(A_1A_2A_3+2B_1B_2B_3-A_1B_1^2-A_2B_2^2-A_3B_3^2)=0\ldots(7).$$

In like manner, p_1, p_2, p_3 are the roots of

$$p^3-(a_1+a_2+a_3)\,p^2+(a_2a_3+a_1a_3+a_1a_2-b_1^2-b_2^2-b_3^2)\,p$$
$$-(a_1a_2a_3+2b_1b_2b_3-a_1b_1^2-a_2b_2^2-a_3b_3^2)=0\ldots(8).$$

Now it is evident from (5), that (λ being properly assumed) if we substitute λP for p in (8), (and divide by λ^3), the resulting equation will be identical with (7); hence equating

the coefficients and eliminating λ, we find the two conditions of similarity to be

$$\left.\begin{aligned} \frac{A_2A_3+A_1A_3+A_1A_2-B_1^2-B_2^2-B_3^2}{(A_1+A_2+A_3)^2} &= \frac{a_2a_3+a_1a_3+a_1a_2-b_1^2-b_2^2-b_3^2}{(a_1+a_2+a_3)^2}\,{}^{*} \\ \frac{A_1A_2A_3+2B_1B_2B_3-A_1B_1^2-A_2B_2^2-A_3B_3^2}{(A_1+A_2+A_3)^3} &= \frac{a_1a_2a_3+2b_1b_2b_3-a_1b_1^2-a_2b_2^2-a_3b_3^2}{(a_1+a_2+a_3)^3} \end{aligned}\right\} \ldots\ldots(9).$$

When the surfaces (1) and (2) are referred to oblique axes, the conditions of similarity (which are of course much more complicated) may be found in the very same manner, the 'discriminating cubic' in this case being

$$\begin{aligned}(P-A_1)(P-A_2)(P-A_3)-(fP-B_1)^2(P-A_1)-(gP-B_2)^2(P-A_2)\\ -(hP-B_3)^2(P-A_3)+2(fP-B_1)(gP-B_2)(hP-B_3)=0\ldots(10),\dagger\end{aligned}$$

where f, g, and h are the cosines of the angles which the axes make with each other.

The analogous condition of similarity for two conics

$$Ax^2+By^2+2Cxy=V\ldots\ldots\ldots\ldots\ldots(11),$$

and

$$ax^2+by^2+2cxy=v\ldots\ldots\ldots\ldots\ldots(12),$$

namely

$$\frac{AB-C^2}{(A+B)^2}=\frac{ab-c^2}{(a+b)^2}\ldots\ldots\ldots\ldots\ldots(13),$$

(the axes being rectangular), is ascribed by Mr. Salmon (*Conic Sections*, p. 204, 2nd edit.) to Mr. Jellett. It may be investigated in the same manner as (9) from the quadratic

$$(P-A)(P-B)-C^2=0\ldots\ldots\ldots\ldots\ldots(14).$$

If the axes be oblique, then, instead of (14), we shall have

$$(P-A)(P-B)-(fP-C)^2=0\ldots\ldots\ldots(15),$$

---

* By deducting each member (multiplied by 2) of the above equation from unity, it will assume the form

$$\frac{A_1^2+A_2^2+A_3^2+2B_1^2+2B_2^2+2B_3^2}{(A_1+A_2+A_3)^2}=\frac{a_1^2+a_2^2+a_3^2+2b_1^2+2b_2^2+2b_3^2}{(a_1+a_2+a_3)^2}.$$

† I am not aware that (10) has been published before; certainly it is not generally known. I intended giving the investigation in my Chapters on "Analytical Geometry of Three Dimensions relative to Oblique Axes," in the *Mathematician*, but the discontinuance of that work prevented the completion of my design. It will be observed that (10) differs from (6) only in having $fP-B_1$, $gP-B_2$, $hP-B_3$, instead of $-B_1$, $-B_2$, $-B_3$.

where f is the cosine of the inclination of the axes; and the condition of similarity will in this case be

$$\frac{AB - C^2}{(A + B - 2fC)^2} = \frac{ab - c^2}{(a + b - 2fc)^2} \text{ .... } (16).$$

In the preceding investigation the two surfaces (or conics) have been supposed referred to the same axes. If, however, they be referred to different axes, the conditions of similarity will evidently remain unchanged providing both systems of axes be rectangular; but if the axes be oblique the conditions, which are easily found from (10) or (15), will be somewhat different; thus (16) will become

$$\frac{(AB - C^2)(1 - f^2)}{(A + B - 2fC)^2} = \frac{(ab - c^2)(1 - f'^2)}{(a + b - 2f'c)^2},$$

or

$$\frac{(AB - C^2)\sin^2 w}{(A + B - 2C\cos w)^2} = \frac{(ab - c^2)\sin^2 w'}{(a + b - 2c\cos w')^2} \text{ ..... } (17),$$

w and w' being the inclinations of the two systems of axes.

York Town, near Bagshot,
Jan. 21, 1852.

EASY METHOD OF FINDING THE MOMENTS OF INERTIA OF AN ELLIPSOID ABOUT ITS PRINCIPAL AXES.

By the late G. W. Hearn.*

The equation to the ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ ....} (1),$$

make $x = ax'$, $y = by'$, and $z = cz'$; then, at the surface,

$$x'^2 + y'^2 + z'^2 = 1 \text{ ....} (2).$$

Let A, B, C be the moments of inertia about the axes of x, y, z, respectively, and ρ the density; therefore

$$A = \rho\iiint(y^2 + z^2)\,dx\,dy\,dz = \rho abc\iiint(b^2y'^2 + c^2z'^2)\,dx'dy'dz'$$
$$= abc\{b^2.\Sigma m'y'^2 + c^2.\Sigma m'z'^2\}\text{...}(3),$$

where $(x'y'z')$ denotes any point within the sphere (2), and m' the element (of the sphere) at that point.

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* Communicated by Mr. Weddle.

Now, in the case of a sphere, we evidently have

$$\Sigma m'x'^2 = \Sigma m'y'^2 = \Sigma m'z'^2 = \tfrac{1}{3}\Sigma m'(x'^2 + y'^2 + z'^2) = \tfrac{1}{3}\Sigma m'r'^2,$$

and $$m' = \rho . 4\pi r'^2 . dr' ;$$

also, (2), the limits of r' are 0 and 1, therefore

$$\Sigma m'r'^2 = 4\pi\rho \int_0^1 r'^4 dr' = \tfrac{4}{5}\pi\rho ;$$

and $$\Sigma m'x'^2 = \Sigma m'y'^2 = \Sigma m'z'^2 = \tfrac{4}{15}\pi\rho.$$

Hence, (3), $A = abc(b^2 + c^2)\tfrac{4}{15}\pi\rho = \tfrac{4}{15}\pi\rho abc(b^2 + c^2)$.

But M, the mass of the ellipsoid, $= \tfrac{4}{3}\pi\rho abc$, therefore

$$A = \tfrac{1}{5}M(b^2 + c^2) ;$$

and B and C may of course be found in a similar manner.

THEOREMS ON COMBINATIONS.

By the Rev. Thomas P. Kirkman, M.A.

It has been observed by Mr. Cayley, somewhere in the *Philosophical Magazine*, that two systems, and only two, of triads can be made with 7 symbols *abcdefg*, so that the systems shall have no common triad, and that each of them shall exhibit once every duad possible with the symbols. From this it follows that

A. *With seven symbols can be formed* 21 *triads, so that every duad possible shall be three times employed.*

These 21 triads are those which remain unemployed in the two systems of seven.

To every triad, *abc*, belongs a complementary quadruplet, *defg*. It is easily seen that

B. *Two systems of seven quadruplets can be made with seven symbols, each system twice exhibiting all the* 21 *duads, and so that the systems shall have no quadruplet in common.*

C. *A system of* 21 *quadruplets can be made with seven symbols, so that every possible duad shall be six times employed.*

These are the 21 quadruplets that remain after the formation of the two systems of seven.

If *abcd*, *abce*, *abcg* are three of these 21, then will *cdeg*, *bdeg*, *adeg* be three others.

With 9 symbols can be made seven different groups, each of twelve triads, every group exhibiting all the duads once. This is not new.

D. *With* 13 *symbols can be made three different groups, each of* 26 *triads, each group containing all the duads once.*

The following,

Aa_1a_2	*abc*	*aab*	*bbc*	*cca*
Aa_3a_4	111	132	133	143
Ab_1b_2	134	241	242	231
Ab_3b_4	223	144	144	134
Ac_1c_2	241	233	231	242
Ac_3c_4	343			
	312			
	424			
	432			

where the second column is $a_1b_1c_1$, $a_1b_3c_4$, &c., is such a group. If 1 and 3 be exchanged, and also *b* and *c*, a second is formed; and a third, by exchanging in the first 1 and 4, and cyclically permuting *cba*. Whether any more such groups can be made without repeating any triad of the three groups, I know no simple method of deciding.

If we write out the duads of 8 things, *hiklmnop*, thus,

hi	*hk*	*hl*	*hm*	*hn*	*ho*	*hp*
ko	*ip*	*ik*	*il*	*im*	*in*	*io*
ln	*lo*	*mo*	*kp*	*kl*	*km*	*kn*
mp	*mn*	*np*	*no*	*op*	*lp*	*lm*

we may prefix to the columns in order the seven letters *abcdefg*, thus forming the triads *ahi*, *ako*,... *bhk*, &c. We can then cyclically permute *abcdefg* in these triads five steps, and thus make five systems, each of 28 triads, in each of which systems all the seven *abcdefg* will once be combined with all the eight *hiklmnop*. If we form with the seven the 21 triads of Theorem A, and add to these three of the five systems of 28 triads, we shall have 15 symbols thrown into triads till every duad has been thrice employed, without repeating a triad. The two remaining systems of 28 form, with the two systems of 7 triads mentioned at the beginning of this paper, two distinct arrangements of 15 things in triads, each exhibiting all the duads once, and having no triad in common. The sum of the constructed triads contains every duad five times.

If we now write out the pairs of the 8 symbols *abcdefgp*, thus,

ab	*ac*	*ad*	*ae*	*af*	*ag*	*ap*
cg	*bp*	*bc*	*bd*	*be*	*bf*	*bg*
df	*dg*	*eg*	*cp*	*cd*	*ce*	*cf*
ep	*ef*	*fp*	*fg*	*gp*	*dp*	*de*

and prefix to the columns in order the two sets of 7 letters

lh, kn, mh, ko, im, ol, in,

thus forming the 56 triads *lab, hab, lcg, hcg,...kac, nac,... oag, lag,* &c.; it is evident that we shall twice combine every one of the 8 *abedefgp* with each of the 7 *hiklmno*, and that we shall have twice employed all the pairs possible with the eight. Also our 56 triads are all distinct from those above formed with the 15 symbols; for we have before made none containing a pair only of the seven *abcdefg*, and of those which we have just made containing *p*, none have occurred before; *lep* and *ldp*, for example, which we have just formed, are new, because *lp* was combined before only with the five letters *fgabc*, and not with *d* or *e*. If we now form two distinct sets of 7 triads with the 7 elements *hiklmno*, each set exhibiting all the duads, we shall have triads completely new, as containing each *three* of these elements: these 14, added to the twice 28 just made, are a system of triads made with the 15 symbols to exhibit every duad twice. These being added to the system before formed which contains every duad five times, make up a system of triads, all different, which contain every duad seven times. The remainder of the possible triads will contain every duad six times. Thus we can add the theorem

E. *With* 15 *symbols triads can be formed, so as to exhaust every possible duad once, twice, three, four, five, six, seven, eight, nine, ten, eleven, twelve, or thirteen times, and so that no triad shall be twice employed, whichever of these numbers we fix upon.*

F. *With* $(12n+3)$ *symbols can be formed triads so as to exhaust the duads* $6n-1$ *or* $6n+2$ *times; and with* $(12n+7)$ *symbols, so as to employ every duad* $6n+1$ *or* $6n+3$ *times.*

This latter theorem is proved by writing all the triads possible with $6n+1$ (or $6n+3$) things, and prefixing $6n-1$ (or $6n+1$) of these things to the duads made with the remaining $6n+2$ (or $6n+4$) things, which can easily be done without repetition of any triad.

From Theorem D follows easily that

G. *With* 27 *things triads can be made till every duad has been either twice or thrice employed.*

Mr. Salmon, discussing, at p. 196 of his *Higher Plane Curves*, the double tangents of curves of the fourth order, has laid bare, by a single stroke of his bright analytic wand, the pleasing property, that 28 things can be thrown into quadruplets till every duad has been five times employed. I find that this is a case of the following theorem:

H. $4(3n+1)$ *things can be arranged in quadruplets till every duad has been* $2n+1$ *times employed.*

To prove this we take $12n+3$ symbols, viz. the three bcd, the $4n$ unaccented letters $efghij\ldots$, the $4n$ accented $e'f'g'h'i'j'\ldots$, and the $4n$ subaccented $e_1f_1g_1h_1i_1j_1\ldots$, and arrange them thus:

$$
\begin{array}{llllllll}
bcd & & & & & & & \\
bef & ce'f' & de_1f_1 & ee'e_1 & ff'f_1 & ge'g_1 & hf'h_1 & ie'i_1 \\
bgh & cg'h' & dg_1h_1 & eg'g_1 & fh'h_1 & gg'i_1 & hh'j_1 & ig'k_1 \\
bij & ci'j' & di_1j_1 & ei'i_1 & fj'j_1 & gi'k_1 & hj'l_1 & \vdots \\
bkl & ck'l' & dk_1l_1 & ek'k_1 & fl'l_1 & gk'm_1 & hl'n_1 & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

Here the second vertical rows of the even columns after the second are always the same $2n$ accented letters, and their third vertical rows are the $2n$ cyclical permutations of the same series subaccented. The second vertical rows of the odd columns after the third are all one accented series, while their third rows are that series subaccented and cyclically permuted, the series being the remaining half of the $4n$ letters.

The number of the columns is $4n+3$, equal to that of the unaccented letters $abcde\ldots$, any one of which heads every triad in one column.

The number of triads in every column, excluding the triplet bcd, is $2n$.

It is evident that each of the $12n+3$ letters occurs $2n+1$ times, and that no duad is twice employed. If then we prefix a to every triad, we shall have quadruplets $abcd$, $abef$, $ace'f'$, &c., in which a is combined $2n+1$ times with every other symbol. Let us suppose that a is thus prefixed throughout.

We next execute the law, that if $\alpha\beta\delta\theta$ and $\alpha\beta\mu\phi$ be two quadruplets, $\delta\theta\mu\phi$ shall be a third. It will thus come to pass that every duad will be employed $2n+1$ times, and no more. For every unaccented duad will occur $2n+1$ times, as *cd* or *ce*. The former occurs in *abcd*, and the $2n$ quadruplets *cdef*, *cdgh*, *cdij*, &c., made by our law from the first column. The second and fourth columns give us $ace'f'$ and $aee'e_1$, whence comes $cef'e_1$; as also $ceh'g_1$ from $acg'h'$ and $aeg'g_1$; and in the same way $2n$ quadruplets containing *ce* can be added to *cdef*.

Also, every duad containing *a* and an accented or subaccented letter, occurs $2n+1$ times, as ae', which is found in $ace'f'$, $ace'e_1$, $age'g_1$, &c., from the $2n+1$ even columns: ae_1 occurs in ade_1f_1, $aee'e_1$, and $2n-1$ times more from the 6th, 8th, 10th, &c. columns: ah_1 occurs in adg_1h_1, $afh'h_1$, &c. made from the third and $2n$ more odd columns. The duad ee_1 occurs once in $aee'e_1$; and if we write under this quadruplet the other $2n$, which contain the duad ae', our law will give us $2n$ more containing ee_1.

The duad $e'f'$ occurs once in $ace'f'$, and $2n$ times more in the quadruplets $e'f'g'h'$, $e'f'i'j'$, &c., made from the second column.

The duad $e'f_1$ does not appear in the quadruplets which contain the letter *a*; but $aee'e_1$ and ade_1f_1, appear, wherefore $ede'f_1$ is a quadruplet, under which, if we write the remaining $2n$ which contain *ed*, we shall deduce by our law $2n$ more containing $e'f_1$. In this manner it can be proved that any duad possible with the $12n+4$ symbols occurs $2n+1$ times, and the Theorem H is established.

I shall conclude with a theorem interesting both in itself and for the simplicity of its demonstration.

J. 2^n *young ladies can all walk out in fours day by day till every three have once walked together.*

First, let n be even and $= 2m$.

Take for the 4^m symbols the m-plets that can be made with the four letters *abcd*. *Every* triplet formed of these symbols is completed in *one* way only into a quadruplet by the simple rule, that *no quadruplet shall have only three* r^{th} *places either like or unlike.*

Thus the triplet (supposing $m=6$)

abbadd . bbcaab . cdaaaa

becomes by this rule

abbadd . bbcaab . cdaaaa . dddadc.

In fact, if three r^{th} places in the triplet are all different letters, the r^{th} place of the fourth symbol is the remaining fourth letter; if they are all one letter, the fourth r^{th} place is that one; if they shew two letters only, the fourth shews the odd one.

Let any of these $4^m.(4^m-1)(4^m-2):(1.2.3.4)$ quadruplets be chosen at random. It will either exhibit, or not, some four e^{th} places (for one or more values of e) comprising all the letters *abcd*. Such values of e in the above quadruplet are the first places and the sixth. Let the four places answering to the least such value of e be suffered to stand; in this case $e=1$, the units' places, *dbac*. Permute now cyclically *abcd* in the four $(e+1)^{\text{th}}$ places: this gives, including the one before us, four 4-plets, having no symbol in common, and all alike in their e^{th} places. In these four permute *abcd* cyclically in the $(e+2)^{\text{th}}$ places, which produces 4^2 quadruplets, having no common symbol: and if the same cyclical permutations be effected in all these in their four $(e+3)^{\text{th}}$ places, there will arise 4^3 such 4-plets. Continuing these permutations of *abcd* in all the m places except the e^{th}, we obtain 4^{m-1} 4-plets, in which the 4^m symbols are exhausted. It is easily seen that every quadruplet in this group of 4^{m-1} will, if thus treated, produce this same group, and nothing more. Thus it is proved that every 4-plet, which exhibits four unlike e^{th} places, determines a group in which all the symbols are exhausted, and can only determine its own group.

Take next a quadruplet in which no four e^{th} places exhibit the four letters. There must be in it four e^{th} places exhibiting a pair, and also four $e_1{}^{\text{th}}$ places exhibiting a pair; for $Aa.Ab.Ab.Aa$ cannot be one of our 4-plets, because the symbol Aa is repeated in it. Let such a quadruplet, for example, be

$$(\ldots abbb)\,(\ldots acdb)\,(\ldots bcdb)\,(\ldots bbbb).$$

We take the lowest values of e and e_1, which in this instance are $e=2$, $e_1=3$, counting from the right. We conceive *abcd* to have rank or magnitude rising in that order: then we exchange the pair in the e^{th} places for the excluded pair, the less for the less and the greater for the greater. This gives two 4-plets, including that before us. Next, in the $e_1{}^{\text{th}}$ places we exchange the pair exhibited for the excluded pair, the less for the less and greater for greater. This makes our two into four 4-plets, having no symbol in common, any one of which, treated in the manner laid down,

will produce all the four. If we now continually cyclically permute *abcd* in all the places except the e^{th} and the e_1^{th}, we shall make our four quadruplets into 4^{m-1}, all different in their elements, and therefore comprising the 4^m symbols.

We have thus proved that every triplet possible with the 4^m elements or m-plets, determines one quadruplet which determines one group of 4^{m-1}. That is, any three young ladies who choose to walk together on any day, determine the arrangement of all the 4^m for that day.

To prove the truth of the theorem when n is $2m+1$, we add to the former 4^m m-plets made with *abcd*, the 4^m m-plets made with abcd. We then join to our rule above given, in which a and a, b and b, &c., are like letters, the following: *no quadruplet shall have only three italic or roman symbols.* This determines, with the preceding rule, the fourth element to be added to any triplet. Thus, if out of 512 young ladies the three (abbd), (*abbb*), (*adaa*), choose to walk together to-day, their companion must be (adac), and this quadruplet determines the arrangement, for the day, of the 512.

For such a 4-plet must either have, or not, both italic and roman symbols. If it has not, let them be all italic. We can, by the preceding argument, form by it a group of 4^{m-1} 4-plets comprising all the italic symbols, and under these we can write the same group in roman letters. This completes the arrangement of the 2.4^m ladies for the day.

If the quadruplet has both italic and roman symbols, it must either have, or not, some four e^{th} places exhibiting four unlike letters, for one or more values of e. Let it have them, and take the least such value; then let the four e^{th} places stand, while the cyclical permutations are effected in every other four r^{th} places. This gives us 4^{m-1} 4-plets having no symbol in common, under which, if we write the group of 4^{m-1}, which differs therefrom only in the exchange of italic and roman letters, we shall have the 2.4^m ladies arranged for the day.

If the quadruplet having both roman and italic letters exhibits no four e^{th} places containing four unlike letters, it must exhibit four e^{th} places containing a pair of unlike letters. Let such a quadruplet be, $m=4$,

abbd . *abbd* . adad . *adad*,

in which the least such value of e is $e=2$. As these tens or second places, b, b, a, a, are all *different*, though only of two names, we can let them stand, and cyclically permute

as before in the other places, thus obtaining 4^{m-1} quadruplets having no symbol in common. Then, putting in the same e^{th} places dd for bb, and cc for aa, less for less, as before, we obtain, by a like course of cyclical permutations in the remaining places, another group of 4^{m-1} 4-plets, which, added to the preceding, completes the arrangement for the day. And it is easily seen that any quadruplet of any of these groups of 2.4^{m-1}, if treated in the manner prescribed, will give rise to its own complete group and to nothing more. Thus the Theorem J is established.

When $n=4$ in J, the triplets combined with any letter a, are those required for the solution of the problem of the 15 young ladies. But I do not see in the case of $n=6$ any step towards a like handling of 63.

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ON A CLASS OF RULED SURFACES.

By the Rev. George Salmon.

The remarks of Mr. Cayley in the last No. on "Ruled Surfaces in General" (vol. VII. p. 171) have led me to examine more particularly the nature of the surface generated by a right line resting on three directrices, which we shall suppose to be curves of the degrees m_1, m_2, m_3.

First, then, it is plain that the directrices are in general multiple lines on the surface of the degrees respectively m_2m_3, m_3m_1, m_1m_2. For through any point on the first curve pass m_2m_3 lines of the system; the intersections, namely, of the two cones having this point for a common vertex, and resting on the curves m_2, m_3.

The degree of the surface is equal to the number of lines of the system which meet an arbitrary right line; it is therefore equal to the number of intersections of the curve m_3 with the ruled surface, having for directrices the arbitrary right line and the curves m_1, m_2; or it is equal to m_3 times the degree of this latter ruled surface. And by a repetition of the same argument it appears that the degree of the surface is $2m_1m_2m_3$.

The intersection of two lines of the system gives rise to a double point on the surface. Let us therefore examine how many other lines of the system can intersect a given

one. And it is evident that this number is equal to that of the intersections of the curve m_3 with the ruled surface whose directrices are the curves m_1, m_2, and a right line resting on both. The degree of this latter ruled surface is found by examining the nature of a plane section through the right line. And it is found that the right line is a multiple line of the degree $m_1m_2 - 1$, and that there are besides in any such plane section $(m_1 - 1)(m_2 - 1)$ lines of the new system. The degree of the latter ruled surface is therefore $2m_1m_2 - m_1 - m_2$, which meets the curve m_3 in points not on the right line, whose number is

$$(2m_1m_2 - m_1 - m_2)\,m_3 - (m_1m_2 - 1) = 2m_1m_2m_3 - m_2m_3 - m_3m_1 - m_1m_2 + 1.$$

If we consider the section of the ruled surface passing through any line of the system, we know that it must be this right line and a curve of the degree $2m_1m_2m_3 - 1$, intersecting the right line in that number of points. And these points are distributed as follows:

$$m_1m_2 - 1, \quad m_2m_3 - 1, \quad m_3m_1 - 1,$$

where the right line meets each of the directrices; the $2m_1m_2m_3 - m_1m_2 - m_2m_3 - m_3m_1 + 1$ points just mentioned, where the right line meets other lines of the system; and the single point of contact of the plane of section with the ruled surface.

It is to be observed that there are a number of right lines of the system which are double lines on the surface; those, namely, which rest twice on one of the directrices, resting also on each of the other two. The number of such lines, resting twice on the curve m_1, is proved, by reasoning similar to that used before, to be m_2m_3 times the degree of the ruled surface generated by a right line resting twice on m_1, and also on an arbitrary right line. But if we consider the section of such a surface through the arbitrary right line, we shall find that there will be $\frac{m_1(m_1-1)}{1.2}$ right lines, besides the arbitrary right line itself, which is a multiple line of the degree h_1, (h_1 being the number of apparent double points of the curve m_1). It follows, then, that the total number of double right lines on the surface is

$$m_2m_3\left\{\frac{m_1(m_1-1)}{1.2} + h_1\right\} + m_3m_1\left\{\frac{m_2(m_2-1)}{1.2} + h_2\right\}$$
$$+ m_1m_2\left\{\frac{m_3(m_3-1)}{1.2} + h_3\right\}.$$

I am not able to tell the order of the double curve in this case; but I may mention that when Mr. Cayley observes (p. 173, near the end) that it does not appear that there is anything to determine x, he overlooks that he has proved that the reciprocal of the skew surface is one of the same order as itself, and consequently that the theory of reciprocal surfaces must afford equations connecting the characteristics of the double curve with those of the ruled surface. When the curve x is simply a double line, the equation in question is

$$m(m-2)(m-4) = 3x(m-4).$$

The proof of this will appear in a paper on Reciprocal Surfaces, which I wrote about three years ago and which I hope soon to publish. But ordinarily I believe that the curve x will include multiple lines of a degree higher than the double (as, for example, in the case discussed in this paper): and I have not examined the effect of such multiple lines in diminishing the cuspidal and double edges of the tangent cones to surfaces.

Trinity College, Dublin,
Sept. 7, 1852.

Note.—I take this opportunity of saying, in reply to Mr. Walton's article in the last Number, that I see no reason why we should not close our controversy on the terms of arbitration proposed by Professor De Morgan, namely that Mr. Gregory's conventions shall be banished from the regions of Algebraic Geometry to those of Geometrical Algebra, where for my part I have no desire to follow them. In fact Mr. Walton does not deny the only point for which I am anxious to contend, viz. that the curvilinear loci obtained by Mr. Gregory's rules have *no geometrical connexion* with the plane curves represented by the same equations. And if this be so, they cannot be expected to throw any light on any difficulties, real or supposed, in the theory of plane curves. I have only to add that I believe Mr. Walton was hasty in asserting (vol. VII. p. 239) that "if $f(x, y) = 0$ be transcendental, a conjugate point not double but single may easily present itself," and that the case of a conjugate point appearing to have a real tangent is explained by observing that such a point results from the union of two or more ordinary conjugate points.

AN ACCOUNT OF SOME TRANSFORMATIONS OF CURVES.

By Andrew S. Hart.

Among the methods of Transformation of Curves mentioned by Mr. Salmon in his treatise on the *Higher Plane Curves*, is the method of inversion which was introduced to the notice of geometers by Dr. Ingram and Mr. Stubbs

in the *Transactions of the Dublin Philosophical Society* for 1843, and which appears worthy of further development.

The inverse curve is formed by substituting $\frac{1}{\rho}$ for ρ in the polar equation of the given curve, or, which is the same thing, substituting $\frac{x}{x^2+y^2}$, $\frac{y}{x^2+y^2}$ for x and y in the equation referred to rectangular coordinates.

Thus, if the equation of a given curve be

$$u_0 + u_1 + u_2 + \&c. + u_n = 0$$

(where u_0 is the absolute term, and u_1, u_2, &c. the aggregates of all the terms of the first, second, &c. degrees), the equation of the inverse curve will be

$$u_0(x^2+y^2)^n + u_1(x^2+y^2)^{n-1} + u_2(x^2+y^2)^{n-2} + \&c. + u_n = 0,$$

an equation of the $2n^{\text{th}}$ degree having three multiple points of the n^{th} degree, viz. the origin and the two circular points at infinity. But if the origin be on the given curve, $u_0 = 0$, and the degree of the inverse curve will be $2n-1$, the origin will still be of the n^{th} degree, and the circular points at infinity of the $(n-1)^{\text{th}}$ degree. If the origin be a multiple point on the given curve, the degree of the inverse curve and also of its circular points will be diminished by the degree of this multiple point. Again, if the given curve be circular (*i.e.* if it pass through the two circular points), the equation of the inverse will be divisible by x^2+y^2, its degree and the degree of the origin will be reduced by two, and the degree of the circular points at infinity by one. If the given curve be bicircular, each of these reductions will be doubled, and so on. Also, since the origin is the inverse of the line at infinity, the inverse of any parabolic curve will have a cusp at the origin.

For example, the inverse of a right line is a circle passing through the origin; but if the origin be on the given line, the inverse is a right line. The inverse of a conic is a bicircular biquadratic having a double point at the origin, but if the origin be on the conic, the inverse is a circular cubic; if the conic be a parabola, the inverse has a cusp at the origin; and if it be a circle, the inverse is also a circle, if the origin be not on the curve: but if the origin be on the given circle, the inverse is still farther reduced and becomes a right line.

The application of these principles to curves of higher degrees is obvious, and I now proceed to some of the principal geometrical relations of inverse curves to one another.

First it is to be observed, that while as a general rule every point on one line has its corresponding point on the inverse line, the origin and the two circular points at infinity are to be excepted, as we have seen that their presence on any curve gives rise not to corresponding points, but to a reduction of degree of the inverse curve.

Secondly, with these exceptions every multiple point corresponds to one of the same nature on the inverse curve, and every contact or intersection of two lines corresponds to a contact or intersection at the same angle of the inverse lines. Hence, also, since a focus is an infinitely small circle having double contact with the curve, its inverse is a focus of the inverse curve, except in the case where the origin is a focus to which correspond in the inverse curve two imaginary cusps at the circular points at infinity. Thus, if the origin be a focus of a conic section, the inverse curve will be a limaçon, which becomes a cardioide when the given curve is a parabola, and the inverse of a circular cubic or bicircular biquadratic, when the origin is a focus, becomes a Cartesian oval.

Thirdly, if a and b be the points inverse to A and B, $ab = \dfrac{AB}{OA.OB}$, O being the origin, and the distance of any point A from a right line is to the distance of the inverse point from the circle inverse to the line measured in the direction of the origin as OA to the diameter of the circle.

Examples of these properties will be found in Mr. Salmon's Treatise and in the volume of Transactions already referred to. I will only add that, as it is proved (*Higher Plane Curves*, p. 177), that confocal circular cubics cut at right angles, it follows by inversion that all confocal bicircular biquadratics (including Cartesian ovals) cut one another orthogonally at their eight points of intersection; in fact, if any of these points be taken as origin, the tangents will be parallel to the asymptotes of the inverse confocal circular cubics, and it is therefore sufficient to prove that they are perpendicular. Also, since a circular cubic has four tangents parallel to its asymptote, which touch at four of the intersections with the confocal cubic, it follows by inversion that there are four circles which touch a bicircular biquadratic at any point O, each of which touches it a second time at the points A, B, C, D, which are four of the intersections of this biquadratic with the confocal passing through O; the three remaining intersections are the intersections of three pairs of circles which pass respectively through OAB, OCD;

OAC, *OBD*; *OAD*, *OBC*; and each pair of these circles cut one another at right angles.

It would be easy to multiply examples. The above may serve to shew the fertility of this method in geometrical results.

Trinity College, Dublin,
July 28, 1852.

ON THE METHOD OF VANISHING GROUPS.

By James Cockle.

[*Concluded from* Vol. vii. p. 118.]

XXVI. If a system of simultaneous and homogeneous equations admit of finite algebraic determination by any process without its being necessary that more than a certain number of undetermined quantities should enter into each of the given equations, I call that number the *explicit* limit of the process as applied to the system. If, *à priori* and independently of the given system, each of the indeterminates involved in it can be represented as a linear and homogeneous function of not *less* than a certain number of other indeterminates, without our results being thereby rendered illusory, I call the latter number the corresponding *implicit* limit of the system.

If a limit can be found, below which the process becomes illusory, we have an *absolute* limit of the process. If we denote by the *order* of an equation the number of indeterminates which it contains, the limit of a system corresponding to any process is the least order under which it is determinable by that process.

In the application of indeterminate processes to the theory of equations, the implicit limit very early forces itself upon our attention. A simple investigation (which shows the importance of examining and illustrates a method of determining that limit) will be found at pages 128—130 of Mr. G. B. Jerrard's *Researches*. Developing Mr. Jerrard's idea and extending its field by his own peculiar resources, Sir W. R. Hamilton, in his *Inquiry*, has afforded us the means of determining the implicit limit of Mr. Jerrard's process in every possible case.

If instead of the implicit limit, we sought no more than the absolute limit of Mr. Jerrard's process, and of all linear

solutions, it would not be difficult to determine it. Adopting Mr. Sylvester's nomenclature and notation (*Journal*, new series, vol. VI. p. 17—18), and also putting every equation of a given system under the form

$$Q_0\lambda^n + Q_1\lambda^{n-1} + Q_2\lambda^{n-2} + \ldots + Q_{n-1}\lambda + Q_n = 0,$$

let us assume that all the quantities Q_0 vanish. Then if for all values of m, from 1 to r inclusive, the quantities Q_m, which are K_m in number, could be made to vanish, we should obtain a linear solution of the system. This evanescence depends upon the solution of the system indicated by the change of k into K, and this again upon the linear solution of the system derived from the last by the change of K_1 and K_r into $'K_1$ and K_r' respectively. By successive reductions we should ultimately be conducted to $L+2$.

The result thus arrived at does not directly furnish us with the implicit limit, inasmuch as we have not shewn how the quantities Q_0 are to be made to vanish. And, for all that appears, the very operations by which equations are obtained in which such relations are satisfied may conduct to illusory results. But, comparing the above with Sir W. R. Hamilton's formulæ, we see that the implicit limit of Mr. Jerrard's process is the absolute limit of all linear solutions, however obtained.

XXVII. The expression $k_1k_2k_3..k_r$ or κ indicates, throughout this paper, that the system given for determination consists of k_1 linear, k_2 quadratic, k_3 cubic,..., and k_r r^ic^ equations. When, in place of k, numbers or letters used as numbers are inserted, commas are placed between them and their meaning ascertained by a reference to their position, as in the notation of Sir W. R. Hamilton and Mr. Sylvester. The evanescence of the formula indicates the finite algebraic solution of the system, or the reduction of its solution to that of equations involving only two indeterminates.

The symbol $(2r; \theta)$ denotes the double operation consisting (1) of the performance of γ_{2r}, and (2) of the grouping the resulting powers two and two together and making each of the r groups thence arising vanish. Whenever we have a vanishing group, we must use it to eliminate an indeterminate from EVERY expression into which it enters. If we call $\xi^{(r)}$ and $\xi^{(r+1)}$ the leading indeterminates of the group p_rp_{r+1}, it will in general be convenient to consider one of the leading indeterminates as eliminated.

I use the symbols j, s, and vg as characteristic of the respective methods of Mr. Jerrard, of Mr. Sylvester, and

of vanishing groups. When the characteristic of a process is prefixed to a bracketed κ, the compound symbol represents the explicit limit of the process as applied to the determination of the system. Thus,

$$j(0, 1, 1) = 5, \quad s(0, 1, 1) = 4, \quad vg(0, 1, 1) = 4.$$

I have given no symbol for the memorable process of Sir W. R. Hamilton; but in the above instance, and in general, the implicit limit of j, as determined by Sir William Hamilton, is given by s. The symbol e will denote ordinary elimination.

XXVIII. If we assume that

$$k_1 = \alpha, \quad k_2 = \beta, \ldots, k_{r-1} = b, \quad k_r = a,$$

the general explicit limit of the pure method of vanishing groups may be represented by either of the identical expressions

$$vg(k_1k_2 \ldots k_r), \quad vg(\kappa), \quad \text{or } vg(\alpha, \beta, \gamma, \delta, \epsilon, \ldots, b, a),$$

of which the last is equal to Υ, as given in IX. and corrected in X. Let w represent either of the identical expressions

$$u\begin{bmatrix} 4, 5, \ldots, r-1, r \\ \delta, \epsilon, \ldots, \quad b, \quad a \end{bmatrix} \text{ or } vg(k_4k_5 \ldots k_{r-1}k_r);$$

and suppose that

$$\Upsilon = \alpha + 2^\beta\phi_\gamma(1, 2, 1, 1, \ldots, w) \ldots\ldots (16),$$

where the quantities $1, 2, \ldots, w$ are $\gamma + 1$ in number, the first $\gamma - 1$ of them being the quantities affected with the negative sign in the original expression for Υ, and -0 being supposed to be added to the $(\gamma - 1)^{\text{th}}$ exponent. Then, as γ passes successively through the values 4, 3, 2, and 1, the quantities included in the last brackets become, in respective succession,

$$(1, 2, 1, 0, w), \quad (1, 2, 0, w), \quad (1, 0, w), \text{ and } (1, w),$$

the singular discontinuous 2 disappearing from the last two expressions. When $\gamma = 0$, the equation

$$\Upsilon = \alpha + 2^\beta w \ldots\ldots\ldots\ldots\ldots (17)$$

gives the explicit limit. It is to be observed that

$$vg(\alpha, \beta) = \alpha + 2^\beta, \quad (w = 1);$$

$$vg(\alpha, \beta, 0^{\epsilon-3}, 1) = \alpha + 2^{\beta+1} = \alpha + 2^\beta(3.2^0 - 1), \quad (w = 2).$$

XXIX. Consider first the system of quadratics

$$u_1 = 0,\ u_2 = 0, \dots ,\ u_{x-1} = 0,\quad u_x = 0,$$

in which

$$vg(0, x) = 2^x \text{ and } s(0, x) = \tfrac{1}{2}(x^2 + x + 2),$$

and consequently for values of x greater than 2 the process s has an advantage over vg. There is a modification (g_2) of vg in which the explicit limit is given by

$$g_2(0, x) = 2^{x-2}.3 \qquad \dots\dots\dots\dots (18),$$

and which may be represented as follows,

$$(2^{x-2};\, \theta)\, u_1,\ (2^{x-3};\, \theta)\, u_2,\ \dots,\ (2^2;\, \theta)\, u_{x-3},\ (2;\, \theta)\, u_{x-2},$$
$$u_{x-1} = f_{x-1}(3) = 0,\quad u_x = f_x(3) = 0.$$

The functions θ vanish by groups, and f by ordinary elimination, in which case $e(0, 2) = 3$. Whenever we arrive at $vg(0, 1) = 2$, we have the ordinary solution of a quadratic, of which the method of vanishing groups is the indeterminate development; e entails upon us the solution of a biquadratic. But for the values $x = 2, 3, \dots, r$, we thereby obtain an advantage over the linear solution.*

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* The scheme of XXXII. enables us to arrive at (18). For if we have x quadratics and

$$v'(0, x) = 2^{x-1} + 2^{x-2} = y,$$

and perform $(2^{x-1};\, \theta),\ (2^{x-2};\, \theta), \dots, (2^2;\, \theta)$ in due order on the first $x-2$ of them, they will be reduced each to the form $f(y - 2^{x-1})$ and, with the above value of y, admit of solution by vg. The last two quadratics will be solved by the remaining two of the grouped quantities (of which

$$2^{x-1} - (2^{x-2} + 2^{x-3} + \dots + 2^2 + 2) \text{ or } 2$$

still remain) by the aid of e.

We may now replace 2^{x-2} by $2^{x-3} + 2^{x-4}$ in the expression for y. And, pursuing this process continuously, we shall arrive at

$$v'(0, x) = y = 2^{x-1} + 2^{x-3} + 2^{x-5} + \dots,$$

the last term being 2 when x is even, and 4 when x is odd. Of course the properties of the functions ψ, &c. would enable us to modify this formula, but I shall not pursue the inquiry.

I would however observe, that in the relation (obtained by any means we please)

$$\psi(0, p) = q,$$

whenever we find that $2^{x-p}q$ is less than $2^{x-2}.3$, we may change (18) into

$$g_2(0, x) = 2^{x-p}q = 2^{x-p}\, \psi(0, p),$$

with an advantage great in proportion to their difference. Thus, one of the results of XXXI. furnishes us with the formula

$$g_2(0, x) = 2^{x-4}.9,$$

the quantity $2^{x-4}.9$ being less than $2^{x-2}.3$ or $2^{x-4}.12$.

I have thought it for the most part unnecessary to encumber the investigations in the text with linear equations. The implicit limit (l)

XXX. A combination of Lagrange's method of multipliers with an extension of the principle of linear solution

of the system $(0, x, ..)$ is always connected with the implicit limit (l') of $(a, x, ..)$ by the relation

$$l + a = l'.$$

The explicit limits in Mr. Jerrard's process are not in general connected by a similar relation. When linear equations are introduced, Mr. Jerrard increases the explicit limit by a number greater than that of the linear equations.

In concluding the subject of quadratics I may observe, that if in XXV. we make

$$v + w + \lambda_1^{-1}m = x, \quad v - w + \lambda_2^{-1}n = y,$$

$$p - \lambda_1^{-2}m^2 - 2\lambda_1^{-1}\lambda_2^{-1}n^2 = r, \quad 2\lambda_1^{-1}n = t,$$

$$q - \lambda_2^{-2}n^2 - 2\lambda_1^{-1}\lambda_2^{-1}m^2 = s, \quad \text{and } 2\lambda_2^{-1}m = u,$$

the system there given is equivalent to

$$x^2 + ty + r = 0,$$

$$y^2 + ux + s = 0;$$

and if U and U' be functions of g undetermined quantities, then, in general, x and y are linear functions of all those quantities, and t and u linear and r and s quadratic functions of the same $g-2$ of them. Various observations (for instance, the transformation

$$x' = x + \tfrac{1}{2}u, \quad y' = y + \tfrac{1}{2}t)$$

arise upon the system last arrived at; but I must now pass on to ulterior objects.

Let U, V, and W denote three homogeneous quadratic functions of five undetermined quantities. Then by XVI. we have, omitting the indices of γ,

$$\gamma_2 U = h_1^2 + h_2^2 + f^2(3) = pq + f^2(3),$$

p and q respectively replacing the p_1 and p_2 of XI. Let r', s', and t be the undetermined quantities in f. We may consider U, V, and W as functions of $pqr's't$; for U already has that form, and V and W may by transformation, or a process of reduction, be made to take it; and, completing squares by means of the terms involving μ and μ', we are at liberty to make

$$V = (ap + bq)^2 + \mu pq + Pp + Qq + R,$$

$$W = (a'p + b'q)^2 + \mu' pq + P'p + Q'q + R'.$$

The three equations $\quad U = 0, \quad V = 0, \quad W = 0,$

will be satisfied whenever the system

$$U = V - \mu U = W - \mu' U = 0$$

is satisfied. And if we make

$$ap + bq + \rho = X, \quad \text{and } a'p + b'q + \rho' = Y,$$

ρ and ρ' may be so determined as to reduce the last two equations of that system to the respective forms

$$X^2 + SY + T = 0 \quad \text{and} \quad Y^2 + S'X + T' = 0,$$

where S, S' are linear, and T, T' quadratic functions of r', s', and t. We may replace the first equation of the system by

$$XY + \nu X + \nu' Y + \nu'' = 0,$$

where ν'' is a quadratic and ν, ν' are linear functions free from X and Y. In three homogeneous functions of four or more indeterminate quantities it may be important to consider two of the variables as distributed in the manner indicated in the last three equations.

gives results not unworthy of notice. Construct out of the given system $x-z$ equations, each of the form

$$\lambda_m u_m + \lambda_n u_n + \dots + \lambda_r u_r = 0 = \Sigma.\lambda u,$$

and let the solution of these equations, together with that of the z equations

$$u_a = 0, \quad u_b = 0, \dots, u_l = 0,$$

involve the solution of the given system of x quadratics. Let the terms included under Σ be $z'+1$ in number, and make $\quad \Sigma_1 \lambda u = A_1 \xi' + A'_1, \dots, \Sigma_{x-z}\lambda u = A_{x-z}\xi' + A'_{x-z}.$

By means of the $x-z$ linear equations

$$A_1 = 0, \quad A_2 = 0, \dots, A_{x-z} = 0,$$

eliminate $x-z$ indeterminates from $A_1', A_2', \dots, A'_{x-z}$. We shall thus have $x-z$ reduced equations each of the form

$$\Sigma'.\lambda u = B\xi'' + B',$$

and if we eliminate $x-z$ other indeterminates by means of $x-z$ equations $B=0$, and continue this operation z' times, we shall ultimately arrive at $x-z$ equations of the form

$$\Sigma^{(z')}.\lambda u = \alpha'\xi'^2 + \alpha''\xi''^2 + \dots + \alpha^{(z')}\xi^{(z')2} + f(y) = 0.$$

By means of the z' ratios of the $z'+1$ quantities λ, let each of the z' quantities $\alpha', \alpha'', \dots, \alpha^{(z')}$ be made to vanish; and, this being done for each of the $x-z$ equations of the form last given, let our results be denoted by

$$\Sigma(\lambda u)_1 = f_1(y), \quad \Sigma(\lambda u)_2 = f_2(y), \dots, \Sigma(\lambda u)_{x-z} = f_{x-z}(y).$$

Let the number of quantities required for the solution of the given system by the present process be $\psi(0, x)$, a yet unknown function of x. Then, from the course of our eliminations, $\quad y = \psi(0, x) - z'(x - z + 1),$

and, from the conditions of the question,

$$y = \psi'(0, x-z), \quad z' = \psi'(0, z) - 1,$$

where ψ' is the known operation best fitted for our purpose. Hence

$$\psi(0, x) = \psi'(0, x-z) + (x-z+1)\,\{\psi'(0, z) - 1\} \dots (19).$$

XXXI. Let $z=1$; then, putting ψ_1 for ψ,

$$\psi_1(0, x) = \psi'(0, x-1) + x;$$

and if we make ψ' and ψ_1 identical, and treat this last as an equation in finite differences, availing ourselves of e in determining the constant, we find

$$\psi_1(0, x) = \tfrac{1}{2}(x^2 + x) \dots\dots\dots\dots (20),$$

which is true for all values of x greater than 1, and has always an advantage over s. In the present instance we may make

$$\Sigma_1 \lambda u = \lambda_1 u_1 - \lambda_2 u_2,\ \Sigma_2 \lambda u = \lambda_1 u_1 - \lambda_3 u_3, \ldots, \Sigma_{x-1} \lambda u = \lambda_1 u_1 - \lambda_{x-1} u_{x\,1},$$

and then solve $u_1 = 0$ by means of ξ'.

This process is equivalent to another which should proceed by causing ξ'^2 to disappear by ordinary elimination, and ξ' by making its coefficient vanish.

In (19) let $z = 2$, then

$$\psi_2(0, x) = \psi'(0, x - 2) + (x - 1).2 \ldots\ldots (21);$$

and, if we make $x = 4$ and ψ' the same as e, we find

$$\psi_2(0, 4) = 3 + 3.2 = 9;\quad \psi_1(0, 4) = 10;$$

and, in this instance, ψ_2 has an advantage over ψ_1, and a still greater one over s. The two Σ functions will be

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0, \text{ and } \mu_1 u_1 + \mu_2 u_2 + \mu_3 u_4 = 0,$$

and the remaining equations to be solved will be

$$u_1 = 0 \text{ and } u_2 = 0.$$

So we should find

$$\psi_2(0, 5) = 6 + 4.2 = 14;\quad \psi_1(0, 5) = 15;\quad s(0, 5) = 16.$$

With this fragment on multilinear solution (as I should propose to call it) I shall leave the subject of simultaneous quadratics, and proceed to systems in which quadratics are combined with *one* higher equation: my object being, not to enter into the general question of the implicit limit of the Method of Vanishing Groups, but to point out cases in which the explicit limit of that method is the implicit limit of that of Mr. Jerrard.

XXXII. To the process exhibited in the following scheme I appropriate the characteristic v_2. It is a modification of the Method of Vanishing Groups. After performing $(r;\theta)$ on f_m, I, as a matter of convenience, place to the right of the result the effect of the eliminations on the functions whose suffixes are greater than m.

$$(2^x;\theta) f_1(z) = f_1(z - 2^x);\ f(z - 2^{x-1}):$$

$$(2^{x-1};\theta) f_2(z - 2^{x-1}) = f_2(z - 2^x);\ f(z - 2^{x-1} - 2^{x-2}):$$

$$(2^{x-2};\theta) f_3(z - x_2') = f_3(z - 2^x);\ f(z - x_3'):$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(2^2;\theta) f_{x-1}(z - x'_{x-2}) = f_{x-1}(z - 2^x);\ f(z - x'_{x-1}):$$

$$(2;\theta) f_x(z - x'_{x-1}) = {}^{\backprime}h_1^2 + {}^{\backprime}h_2^2 + f_x(z - 2^x);$$

in which scheme x_m' represents

$$2^{x-1} + 2^{x-2} + \dots + 2^{x-m+1} + 2^{x-m},$$

and consequently $x'_{x-1} = 2^x - 2$.

XXXIII. The explicit limit of this process is

$$v_2(0, x, 0^{s-3}, 1) = 2^x + \psi(x) = z,$$

where ψ is any of the functions which we have used in the theory of quadratics. For, the determination of the system is, by the scheme of XXXII., reduced to the solution of x quadratics, each of the form

$$f(z - 2^x) = f\{\psi(x)\} = 0,$$

and which consequently admit of solution. These being solved, we have only to eliminate an indeterminate between the linear $${}'h_1 \pm {}'h_2 \sqrt{(-1)} = 0,$$

and the given s^ic equation, and the system is determined. When $x = 2$ we have

$$v_2(0, 2, 0^{s-3}, 1) = 2^2 + e(2) = 7 = s(0, 2, 0^{s-3}, 1).$$

For $x = 1$ we have

$$vg(0, 1, 0^{s-3}, 1) = 2^2 = 4 = s(0, 1, 0^{s-3}, 1).$$

In both these cases our explicit limit is the implicit limit of Mr. Jerrard's process.

XXXIV. The combination of the theory of linear transformations with that of vanishing groups gives one or two interesting results. Let a solution of the system

$$0, 2, 0^{s-3}, 1 = 0,$$

be required, and let z be the explicit limit. Then, using f to denote the quadratics, we have, as in XXXII.,

$$(2\,;\,\theta)\, f_1(z) = f_1(z - 2)\,;\, f(z - 1):$$
$$(2\,;\,\theta)\, f_2(z - 1) = f_2(z - 3):$$

and, if $z = 7$, we have, without the aid of equations higher than biquadratics, the following linear transformations,

$$f_2(z - 3) = f_2(4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2,$$
$$f_1(z - 2) = f_1(5) = a_1^2x_1^2 + a_2^2x_2^2 + a_3^2x_3^2 + a_4^2x_4^2 + a_5^2x_5^2;$$

whence f_1 and f_2, under the last form, are satisfied by substituting $-\Sigma_4(x^2)$ for x_5^2 in $f_1(5)$, and solving the result by vanishing groups, that is to say, by breaking it up into two

equations of the form

$$b_1^2x_1^2 + b_2^2x_2^2 = 0, \quad b_3^2x_3^2 + b_4^2x_4^2 = 0.$$

After all the eliminations there will yet remain one disposable quantity in the final s^ic, by means of which it may be solved.

XXXV. A second result deduced by the aid of linear transformation is the following. Let

$$0, 3, 0^{s-1}, 1 = 0,$$

be the system for determination, and let z be the explicit limit. Proceed as follows:

$$(2^3; \theta) f_1(z) = f_1'(z - 2^3); f(z - 2^2):$$

$$\gamma_4 f_2(z - 2^2) = h_1^2 + h_2^2 + h_3^2 + h_4^2 + f_2'(z - 2^3);$$

make $$f_1'(z - 2^3) = 0 \text{ and } f_2'(z - 2^3) = 0,$$

which can be done by ordinary elimination and the solution of a biquadratic, provided that

$$z - 2^3 = 3, \quad \text{or } z = 11.$$

The solution of the given system is now reduced to that of the given s^ic, of

$$h_1^2 + h_2^2 + h_3^2 + h_4^2 = 0 = \Sigma_4(h^2),$$

and of $$f_3(z - 2^2 - 2) = 0 = f_3(z - 6);$$

the last 2 being subtracted in f_3 to indicate that two of the quantities involved in that function are determined by the solution of the equations $f' = 0$.

Now $f_3(z - 6)$ or $f_3(5)$ may be represented as a quadratic function of five indeterminates $h_1, h_2, \ldots, h_5$. And, as in XXXIV., we may, with the aid of a biquadratic, pass, by linear transformation, from the system

$$\Sigma_4(h^2) = 0, \quad f_3(5) = 0,$$

to the other system

$$\Sigma_5(x^2) = 0, \quad \Sigma_5(a^2x^2) = 0;$$

and solving this last system by ordinary elimination and vanishing groups, as in XXXIV., we shall find one undetermined ratio in the final s^ic; in other words, an s^ic determination of the given system. The explicit limit of this combined process is the implicit limit of Mr. Jerrard's process as applied to the system last discussed, for

$$s(0, 3, 0^{s-3}, 1) = 11.$$

XXXVI. There are other perhaps not less important applications of linear transformation to the theory of equations. In fact it is not difficult to see, that three simultaneous homogeneous quadratics involving five unknowns admit of finite algebraic solution without demanding the solution of an equation higher than an equation of the fifth degree. For, by known processes, two of the given equations may be simultaneously transformed into pure homogeneous quadratics involving five new unknowns linearly connected with the given ones. Let $x_1, x_2, \ldots, x_5$ be the new quantities, then the transformed quadratics may be written thus,

$$\Sigma.x^2 = 0,\quad \Sigma.ax^2 = 0,\quad \Sigma.bx^2 + \Sigma.cx_1x_2 + x_5\Sigma.dx = 0,$$

where the last Σ only includes four terms, or

$$\Sigma.dx = d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 = D.$$

Now if by means of $D = 0$ we eliminate x_4 from the transformed system, we shall have three resulting quadratics in x_1, x_2, x_3, and x_5, which, when respectively divided by x_5^2, may be made to form a system of pseudo-homogeneous quadratics completely resolvible by a process similar to that by which Frend solved the problem known as Colonel Titus's. This process entails upon us the necessity of solving a biquadratic: by means of others, that need not be further adverted to, we may avoid the occurrence of an equation higher than a cubic.

XXXVII. The whole indeterminate department of the theory of equations has a peculiar interest attached to it, inasmuch as it suggests an inquiry into the extent to which the limits supposed to be affixed to the solution of systems of simultaneous equations by the researches of Sir W. R. Hamilton and Mr. Sylvester may be legitimately considered as impassable. So far as their views have yet been developed they do not seem to include such cases as those discussed in XXX. and XXXI., which appear to have burst through the bounds prescribed in their investigations. However this be, I shall not now attempt to enter into the question. But, in concluding this series of papers, while I owe an apology for the imperfect manner in which the subject has been handled, I may be allowed to express a hope that the general indeterminate method here discussed will be found, from its simplicity of conception and uniformity of process, to admit of being easily grasped in its most general form, while in its application to more elementary cases (the transformation of the general equation of the fifth degree to the trinomial

form, for example) it possesses the great advantage of having its explicit limit identical with the implicit one of linear solution, while its processes are all purely algebraic. The next great question will be, how far the indeterminate processes are limited to obtaining linear solutions, and whether another range may not be given to them. But this question I must leave.

2, *Pump Court, Temple,*
October 15, 1852.

Erratum.—Art. XII. line 8, *for* $+\rho_s$ *read* $-\rho_s$.

ON A THEOREM CONCERNING THE COMBINATION OF DETERMINANTS.

By J. J. Sylvester, F.R.S.

Let ${}^1A$ represent the line of terms ${}^1a_1, {}^1a_2, \dots {}^1a_m$,
${}^1B$ . . . . ${}^1b_1, {}^1b_2, \dots {}^1b_m$.

Let ${}^1A \times {}^1B$ represent $\Sigma\,({}^1a_r \times {}^1b_r)$, where of course there are r terms within the symbol of summation.

Again, let ${}^2A$ represent the line ${}^2a_1, {}^2a_2, \dots {}^2a_m$,
${}^2B$ . . . . ${}^2b_1, {}^2b_2, \dots {}^2b_m$,

and let $\begin{matrix} {}^1A \\ {}^2A \end{matrix} \times \begin{matrix} {}^1B \\ {}^2B \end{matrix}$ represent $\Sigma\left\{\begin{vmatrix} {}^1a_r, & {}^1a_s \\ {}^2a_r, & {}^2a_s \end{vmatrix} \times \begin{vmatrix} {}^1b_r, & {}^1b_s \\ {}^2b_r, & {}^2b_s \end{vmatrix}\right\}$,

$\begin{vmatrix} {}^1a_r, & {}^1a_s \\ {}^2a_r, & {}^2a_s \end{vmatrix}$ denoting the determinant $({}^1a_r.{}^2a_s - {}^1a_s.{}^2a_r)$,

$\begin{vmatrix} {}^1b_r, & {}^1b_s \\ {}^2b_r, & {}^2b_s \end{vmatrix}$ . . . . $({}^1b_r.{}^2b_s - {}^1b_s.{}^2b_r)$,

there will of course be $m.\frac{1}{2}(m-1)$ terms comprised within the sign of summation; and so, in general, let

$$\begin{vmatrix} {}^1A \\ {}^2A \\ {}^3A \\ \vdots \\ {}^mA \end{vmatrix} \times \begin{vmatrix} {}^1B \\ {}^2B \\ {}^3B \\ \vdots \\ {}^mB \end{vmatrix} \quad m \text{ being less than } n,$$

$\left(\begin{array}{l}\text{and where in general } {}^rA \text{ denotes } {}^ra_1, {}^ra_2, \ldots {}^ra_n \\ \qquad\quad \text{and } {}^rB \text{ denotes } {}^rb_1, {}^rb_2, \ldots {}^rb_n\end{array}\right)$ represent

$$\Sigma \left\{ \begin{vmatrix} {}^1a_{h_1}, & {}^1a_{h_2}, \ldots & {}^1a_{h_m} \\ {}^2a_{h_1}, & {}^2a_{h_2}, \ldots & {}^2a_{h_m} \\ \ldots & \ldots & \ldots \\ {}^ma_{h_1}, & {}^ma_{h_2}, \ldots & {}^ma_{h_m} \end{vmatrix} \times \begin{vmatrix} {}^1b_{h_1}, & {}^1b_{h_2}, \ldots & {}^1b_{h_m} \\ {}^2b_{h_1}, & {}^2b_{h_2}, \ldots & {}^2b_{h_m} \\ \ldots & \ldots & \ldots \\ {}^mb_{h_1}, & {}^mb_{h_2}, \ldots & {}^mb_{h_m} \end{vmatrix} \right\}$$

Now let (r) be any integer less than (m), and let

$$\mu = \frac{m.(m-1)\ldots(m-r+1)}{1.2\ldots r},$$

and let $G_1, G_2, \ldots G_\mu$ denote the μ rectangular matrices of the forms

$$\left.\begin{vmatrix} A_{\theta_1} \\ A_{\theta_2} \\ \ldots\ldots \\ A_{\theta_r} \end{vmatrix}\right\} \text{ respectively,}$$

and let $H_1, H_2, \ldots H_\mu$ denote the μ rectangular matrices of the forms

$$\left.\begin{vmatrix} B_{\theta_1} \\ B_{\theta_2} \\ \ldots\ldots \\ B_{\theta_r} \end{vmatrix}\right\} \text{ respectively.}$$

Now form the determinant

$$\begin{array}{lll} G_1 \times H_1; & G_1 \times H_2 \ldots; & G_1 \times H_\mu; \\ G_2 \times H_1; & G_2 \times H_2 \ldots; & G_2 \times H_\mu; \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ G_\mu \times H_1; & G_\mu \times H_2 \ldots; & G_\mu \times H_\mu; \end{array}$$

then, if we give r the successive values $1, 2, 3 \ldots m$, (in which last case the determinant in question reduces to a single term), the values of the determinant above written will be severally in the proportions of

$$K, K^m, K^{\frac{1}{2}m(m-1)}, \ldots K^m, K;$$

that is to say, the logarithms of these several determinants will be as the coefficients of the binomial expansion $(1+x)^m$.

When we make $r=m$, and equate the determinant corresponding to this value of r with that formed by making $r=1$, the theorem becomes identical with a theorem previously given by M. Cauchy, for the Product of Rectangular Matrixes.

It would be tedious to set forth the demonstration of the general theorem in detail. Suffice it here to say that it is a direct corollary from the formula marked (4) in my paper in the *Philosophical Magazine* for April 1851, entitled "On the Relations between the Minor Determinants of Linearly Equivalent Quadratic Functions," when that formula is particularized by making

$$\left.\begin{matrix} a_{m+1}, & a_{m+2}, & \dots & a_{m+n} \\ b_{m+1}, & b_{m+2}, & \dots & b_{m+n} \end{matrix}\right\}$$

represent a determinant all whose terms are zeros except those which lie in one of the diagonals, these latter being all units, which comes, in fact to defining that

$$\begin{vmatrix} a_{m+e} \\ b_{m+e} \end{vmatrix} = 1, \quad \text{and} \quad \begin{vmatrix} a_{m+\epsilon} \\ b_{m+\epsilon} \end{vmatrix} = 0.$$

The important theorem here referred to is made almost unintelligible by an unfortunate misprint of ${}^{q}\theta_m$, ${}^{1}\theta_m$, ${}^{2}\theta_m$, ${}^{\mu}\theta_m$, in place of ${}^{q}\theta_r$, ${}^{1}\theta_r$, ${}^{2}\theta_r$, ${}^{\mu}\theta_r$. I may here take notice of another and still more inexplicable blunder in the same paper, formula (3), in the latter part of the equation belonging to which

$$\begin{Bmatrix} a_{\theta_1}, & a_{\theta_2}, & \dots & a_{\theta_m}, & a_{\theta_{m+1}}, & a_{\theta_{m+2}}, & \dots & a_{\theta_{m+s}} \\ a_{\phi_1}, & a_{\phi_2}, & \dots & a_{\phi_m}, & a_{\phi_{m+1}}, & a_{\phi_{m+2}}, & \dots & a_{\phi_{m+s}} \end{Bmatrix}$$

is written in lieu of

$$\begin{Bmatrix} a_1, & a_2, & \dots & a_m, & a_{\theta_{m+1}}, & a_{\theta_{m+2}}, & \dots & a_{\theta_{m+s}}.a_{n+1}.a_{n+2}\dots a_{n+m} \\ a_1, & a_2, & \dots & a_m, & a_{\phi_{m+1}}, & a_{\phi_{m+2}}, & \dots & a_{\phi_{m+s}}.a_{n+1}.a_{n+2}\dots a_{n+m} \end{Bmatrix}.$$

7, *New Square, Lincoln's Inn,*
December 16, 1852.

NOTE ON THE CALCULUS OF FORMS.

By J. J. Sylvester, F.R.S.

Accidental causes have prevented me from composing the additional sections on the Calculus of Forms, which I had destined for the present Number of the *Journal*. In the meanwhile the subject has not remained stationary. Among the principal recent advances may be mentioned the following.

1. The discovery of Combinants; that is to say, of concomitants to systems of functions remaining invariable, not only when combinations of the variables are substituted for the variables, but also when combinations of the functions are substituted for the functions; and as a remarkable first-fruit of this new theory of double invariability, the representation of the Resultant of any three quadratic functions under the form of the square of a certain combinantive sextic invariant added to another combinant which is itself a biquadratic function of 10 cubic invariants. When the three quadratic functions are derived from the same cubic function, this expression merges in M. Aronhold's for the discriminant of the cubic. The theory of combinants naturally leads to the theory of invariability for non-linear substitutions, and I have already made a successful advance in this new direction.

2. The unexpected and surprising discovery of a quadratic covariant to any homogeneous function in x, y of the n^{th} degree, containing $(n-1)$ variables cogredient with x^{n-2}, $x^{n-3}y \ldots y^{n-2}$ and possessing the property of indicating *the number of real and imaginary roots* in the given function. This covariant, on substituting for the $(n-1)$ variables the combinations of the powers of x, y with which they are cogredient, becomes the Hessian of the given function.*

3. The demonstration due to M. Hermite of a law of reciprocity connecting the degree or degrees of any function or system of functions with the order or orders of the invariants belonging to the system. The theorem itself was first propounded by me about a twelvemonth back, and communicated to Messrs. Cayley, Polignac, and Hermite, as serving to connect together certain phenomena which had presented themselves to me in the theory: unfortunately it appeared to contradict another law too hastily

* This covariant furnishes, if we please, functions symmetrical in respect to the two ends of an equation for determining the number of its real and imaginary roots. The ordinary Sturmian functions, it is well known, have not this symmetry. As another example of the successful application of the new methods to subjects which have been long before the mathematical world and supposed to be exhausted, I may notice that I obtain without an effort, by their aid, a much more simple, practical, and complete solution of the question of the simultaneous transformation of two quadratic functions, or the orthogonal transformation of one such function, than any previously given, even by the great masters Cauchy and Jacobi, who have treated this question.

assumed by myself and others as probably true, and I consequently laid aside the consideration of this great law of reciprocality. To M. Hermite, therefore, belongs the honour of reviving and establishing,—to myself whatever lower degree of credit may attach to suggesting and originating,—this theorem of numerical reciprocity, destined probably to become the corner-stone of the first part of our new calculus; that part, I mean, which relates to the generation and affinities of forms.*

4. I may notice that the Calculus of Forms may now with correctness be termed the Calculus of Invariants, by virtue of the important observation that every concomitant of a given form or system of forms may be regarded as an invariant of the given system and of an absolute form or system of absolute forms combined with the given form or system. As regards that particular branch of the theory of invariants which relates to resultants, or, in other words, to the doctrine of elimination, I may here state the theorem alluded to in a preceding Number of the *Journal,* to wit that if R be the resultant of a system of (n) homogeneous functions of (n) variables, written out in their complete and most general form (so that by definition $R=0$ is the condition that the equations got by making the (n) given functions zero, shall be simultaneously satisfiable by one system of ratios), then the condition that these equations may be satisfied by ι distinct systems of ratios between the (n) variables is $\delta^{\iota}.R=0$, the variation δ being taken in respect to every constant entering into each of the (n) equations.

7, *New Square, Lincoln's Inn,*
January 1853.

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* This theorem of numerical reciprocity promises to play as great a part in the Theory of Forms as Legendre's celebrated theorem of reciprocity in that of Numbers. Another demonstration of it, which leaves nothing to be desired for beauty and simplicity, has been since discovered by Mr. Cayley, which ultimately rests upon that simple law (essentially although not on the face of it a law of reciprocity) given by Euler, which affirms that the number of modes in which a number admits of being partitioned is the same whether the condition imposed upon the mode of partitionment be that no part shall exceed a given number, or that the number of parts constituting any one partition shall not exceed the same number.

ON THE TRIGONOMETRY OF THE PARABOLA.

By the Rev. J. Booth, F.R.S.

From the *Philosophical Transactions*, for 1852. Part II. p. 385.

A fundamental theorem in the theory of elliptic integrals is

$$\cos\omega = \cos\phi\cos\chi - \sin\phi\sin\chi\sqrt{(1 - i^2\sin^2\omega)} \ldots (338).$$

The angles ϕ, χ, ω may be called conjugate amplitudes.

When the hyperconic section is a circle, $i = 0$, and $\cos\omega = \cos\phi\cos\chi - \sin\phi\sin\chi$, whence $\omega = \phi + \chi$, or the conjugate amplitudes are $\phi + \chi$, ϕ and χ. The development of this expression is the foundation of circular trigonometry.

When the hyperconic section is a parabola, $i = 1$, and (338) may be reduced to

$$\tan\omega = \tan\phi\sec\chi + \tan\chi\sec\phi \ldots\ldots\ldots (339).$$

If we make the imaginary transformations,

$$\left.\begin{array}{l} \tan\omega = \sqrt{(-1)}\sin\omega', \ \tan\phi = \sqrt{(-1)}\sin\phi', \\ \tan\chi = \sqrt{(-1)}\sin\chi', \ \sec\phi = \cos\phi', \ \sec\chi = \cos\chi' \end{array}\right\} \ldots (340).$$

The preceding formula will become, on substituting these values, and dividing by $\sqrt{(-1)}$,

$$\sin\omega' = \sin\phi'\cos\chi' + \sin\chi'\cos\phi',$$

the well-known trigonometrical expression for the sine of the sum of two circular arcs.

Hence, by the aid of imaginary transformations, we may interchangeably permute the formulæ of the trigonometry of the circle with those of the trigonometry of the parabola. In the trigonometry of the circle, $\omega = \phi + \chi$, and in the trigonometry of the parabola ω is such a function of the angles ϕ and χ, as will render $\tan[(\phi, \chi)] = \tan\phi\sec\chi + \tan\chi\sec\phi$. We must adopt some appropriate notation to represent this function. Let the function (ϕ, χ) be written $\phi \perp \chi$, so that $\tan(\phi \perp \chi) = \tan\phi\sec\chi + \tan\chi\sec\phi$. This must be taken as the *definition* of the function $\phi \perp \chi$.

In like manner, we may represent by $\tan(\phi \top \chi)$ the function $\tan\phi\sec\chi - \tan\chi\sec\phi$.

In applying the imaginary transformations, or while $\tan\phi$ is changed into $\sqrt{(-1)}\sin\phi$, $\sec\phi$ into $\cos\phi$, and $\cot\phi$ into $-\sqrt{(-1)}\operatorname{cosec}\phi$, $\perp$ must be changed into $+$ and $\top$ into $-$.

$\perp$ and $\top$ may be called logarithmic plus and minus. As examples of the analogy which exists between the trigonometry of the parabola and that of the circle, the following expressions in parallel columns are given; premising that the formulæ, marked by corresponding letters, may be derived singly, one from the other, by the help of the preceding imaginary transformations.

Trigonometry of the Parabola.

$\tan(\phi \perp \chi) = \tan\phi \sec\chi + \tan\chi \sec\phi \ldots\ldots (\alpha.)$

$\tan(\phi \top \chi) = \tan\phi \sec\chi - \tan\chi \sec\phi \ldots\ldots (\beta.)$

$\sec(\phi \underset{\top}{\perp} \chi) = \sec\phi \sec\chi \pm \tan\phi \tan\chi \ldots\ldots (\gamma.)$

$$\sin(\phi \perp \chi) = \frac{\sin\phi + \sin\chi}{1 + \sin\phi \sin\chi} \ldots\ldots (\delta.)$$

$$\sin(\phi \top \chi) = \frac{\sin\phi - \sin\chi}{1 - \sin\phi \sin\chi} \ldots\ldots (\varepsilon.)$$

Let $\phi = \chi$.

$\tan(\phi \perp \phi) = 2\tan\phi \sec\phi \ldots\ldots (\eta.)$

$\sec(\phi \perp \phi) = \sec^2\phi + \tan^2\phi \ldots\ldots (\theta.)$

$$\sin(\phi \perp \phi) = \frac{2\sin\phi}{1 + \sin^2\phi} \ldots\ldots (\iota.)$$

$$\sec\phi = \frac{e^{\int\frac{d\phi}{\cos\phi}} + e^{-\int\frac{d\phi}{\cos\phi}}}{2}, \quad \tan\phi = \frac{e^{\int\frac{d\phi}{\cos\phi}} - e^{-\int\frac{d\phi}{\cos\phi}}}{2} \ldots\ldots (\kappa.)$$

$1 + \sqrt{(-1)}\tan(\phi \perp \phi) = \{\sec\phi + \sqrt{(-1)}\tan\phi\}^2 \ldots\ldots (\lambda.)$

$$\tan^2\phi = \frac{\sec(\phi \perp \phi) - 1}{2} \ldots\ldots (\mu.)$$

Let the amplitudes be $\phi \perp \chi$ and $\phi \top \chi$.

$\tan(\phi \perp \chi)\tan(\phi \top \chi) = \tan^2\phi - \tan^2\chi \ldots\ldots (\nu.)$

Trigonometry of the Circle. . . (341).

$\sin(\phi + \chi) = \sin\phi \cos\chi + \sin\chi \cos\phi \ldots\ldots$ (a.)

$\sin(\phi - \chi) = \sin\phi \cos\chi - \sin\chi \cos\phi \ldots\ldots$ (b.)

$\cos(\phi \pm \chi) = \cos\phi \cos\chi \mp \sin\phi \sin\chi \ldots\ldots$ (c.)

$$\tan(\phi + \chi) = \frac{\tan\phi + \tan\chi}{1 - \tan\phi \tan\chi} \ldots\ldots \text{(d.)}$$

$$\tan(\phi - \chi) = \frac{\tan\phi - \tan\chi}{1 + \tan\phi \tan\chi} \ldots\ldots \text{(e.)}$$

Let $\phi = \chi$.

$\sin 2\phi = 2\sin\phi \cos\phi \ldots\ldots$ (ē.)

$\cos 2\phi = \cos^2\phi - \sin^2\phi \ldots\ldots$ (th.)

$$\tan 2\phi = \frac{2\tan\phi}{1 - \tan^2\phi} \ldots\ldots (i.)$$

$$\cos\phi = \frac{e^{\phi\sqrt{(-1)}} + e^{-\phi\sqrt{(-1)}}}{2}, \quad \sin\phi = \frac{e^{\phi\sqrt{(-1)}} - e^{-\phi\sqrt{(-1)}}}{2\sqrt{(-1)}} \ldots\ldots \text{(k.)}$$

$1 + \sin 2\phi = (\cos\phi + \sin\phi)^2 \ldots\ldots$ (l.)

$$\sin^2\phi = \frac{1 - \cos 2\phi}{2} \ldots\ldots \text{(m.)}$$

Let the amplitudes be $\phi + \chi$ and $\phi - \chi$.

$\sin(\phi + \chi)\sin(\phi - \chi) = \sin^2\phi - \sin^2\chi \ldots\ldots$ (n.)

Since

$$\sec(\phi \perp \phi) = \sec^2\phi + \tan^2\phi, \text{ and } \tan(\phi \perp \phi) = 2\tan\phi\sec\phi,$$

$$\sec(\phi \perp \phi) + \tan(\phi \perp \phi) = (\sec\phi + \tan\phi)^2.$$

Again, as

$$\sec(\phi \perp \phi \perp \phi) = \sec(\phi \perp \phi)\sec\phi + \tan(\phi \perp \phi)\tan\phi,$$

and $\tan(\phi \perp \phi \perp \phi) = \tan(\phi \perp \phi)\sec\phi + \sec(\phi \perp \phi)\tan\phi,$

it follows that

$$\sec(\phi \perp \phi \perp \phi) + \tan(\phi \perp \phi \perp \phi) = (\sec\phi + \tan\phi)^3,$$

and so on to any number of angles. Hence

$$\sec(\phi \perp \phi \perp \phi \ldots \text{to } n\phi) + \tan(\phi \perp \phi \perp \phi \ldots \text{to } n\phi) = (\sec\phi + \tan\phi)^n \ldots (342).$$

Introduce into the last expression the imaginary transformation $\tan\phi = \sqrt{(-1)}\sin\phi$, and we get Demoivre's imaginary theorem for the circle,

$$\cos n\phi + \sqrt{(-1)}\sin n\phi = \{\cos\phi + \sqrt{(-1)}\sin\phi\}^n.$$

Let $\bar{\omega}$ be conjugate to ψ and ω, while ω, as before, is conjugate to ϕ and χ. Then we shall have

$$\tan\bar{\omega} = \tan(\phi \perp \chi \perp \psi), \text{ or}$$

$$\tan(\phi \perp \chi \perp \psi) = \tan\phi\sec\chi\sec\psi + \tan\chi\sec\psi\sec\phi$$
$$+ \tan\psi\sec\phi\sec\chi + \tan\phi\tan\chi\tan\psi \ldots\ldots (\varpi.)$$

$$\sec(\phi \perp \chi \perp \psi) = \sec\phi\sec\chi\sec\psi + \sec\phi\tan\chi\tan\psi$$
$$+ \sec\chi\tan\psi\tan\phi + \sec\psi\tan\phi\tan\chi \ldots\ldots (\rho.)$$

$$\text{and } \sin(\phi \perp \chi \perp \psi) = \frac{\sin\phi + \sin\chi + \sin\psi + \sin\phi\sin\chi\sin\psi}{1 + \sin\chi\sin\psi + \sin\psi\sin\phi + \sin\phi\sin\chi} \ldots (\sigma.)$$

whence, in the trigonometry of the circle,

$$\sin(\phi + \chi + \psi) = \sin\phi\cos\chi\cos\psi + \sin\chi\cos\psi\cos\phi$$
$$+ \sin\psi\cos\phi\cos\chi - \sin\phi\sin\chi\sin\psi \ldots\ldots\ldots (\text{p.})$$

$$\cos(\phi + \chi + \psi) = \cos\phi\cos\chi\cos\psi - \cos\phi\sin\chi\sin\psi$$
$$- \cos\chi\sin\psi\sin\phi - \cos\psi\sin\phi\sin\chi \ldots\ldots\ldots (\text{r.})$$

$$\tan(\phi + \chi + \psi) = \frac{\tan\phi + \tan\chi + \tan\psi - \tan\phi\tan\chi\tan\psi}{1 - \tan\chi\tan\psi - \tan\psi\tan\phi - \tan\phi\tan\chi} \ldots (\text{s.})$$

Let $(k.\omega)$, $(k.\phi)$, $(k.\chi)$ denote three parabolic arcs, measured from the vertex of the parabola whose parameter is k.

The normal angles of these arcs are ω, ϕ, and χ; ω, ϕ, and χ being conjugate amplitudes. Then

$$2(k.\phi) = k\tan\phi\sec\phi + k\int\frac{d\phi}{\cos\phi}, \quad 2(k.\chi) = k\tan\chi\sec\chi + k\int\frac{d\chi}{\cos\chi},$$

$$2(k.\omega) = k\tan\omega\sec\omega + k\int\frac{d\omega}{\cos\omega};$$

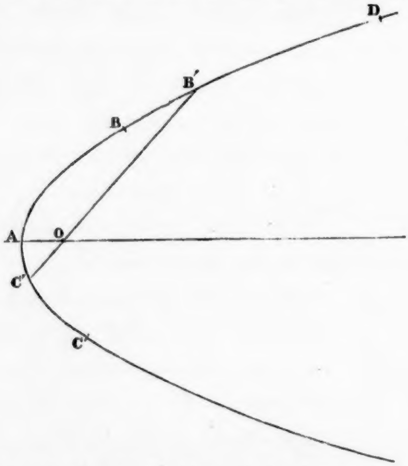
whence, since $\int\frac{d\omega}{\cos\omega} - \int\frac{d\phi}{\cos\phi} - \int\frac{d\chi}{\cos\chi} = 0$, because ω, ϕ, and χ are conjugate amplitudes,

$$(k.\omega) - (k.\phi) - (k.\chi) = k\tan\omega\tan\phi\tan\chi \ldots . (343).$$

Let y, y', y'' be the ordinates of the arcs $(k.\phi)$, $(k.\chi)$, and $(k.\omega)$. Then $y = k\tan\phi$, $y' = k\tan\chi$, $y'' = k\tan\omega$, and the last expression becomes

$$(k.\omega) - (k.\phi) - (k.\chi) = \frac{yy'y''}{k^2} \ldots\ldots\ldots (344).$$

If we call an arc measured from the vertex of a parabola an *apsidal* arc, to distinguish it from an arc taken anywhere along the parabola, the preceding theorem will enable us to express an arc of a parabola, taken anywhere along the curve, as the sum or difference of an apsidal arc and a right line.



Thus let ACD be a parabola, O its focus and A its vertex. Let $AB = (k.\phi)$, $AC = (k.\chi)$, $AD = (k.\omega)$ and $\frac{yy'y''}{k^2} = h$.

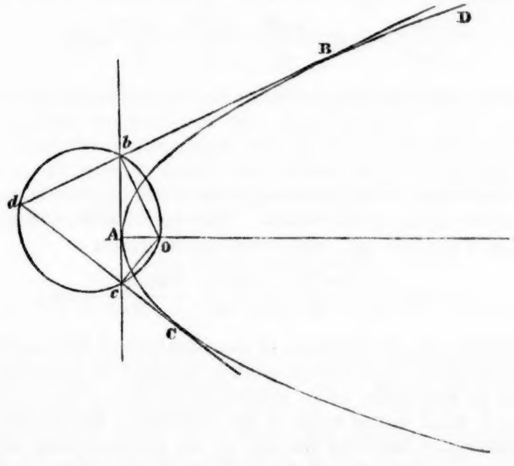
Then (343) shews that the parabolic arc $(AC+AB)$ = apsidal arc $AD-h$; and the parabolic arc $(AD-AB)=BD$ = apsidal arc $AC+h$. When the arcs AC', AB' together constitute a focal arc, or an arc whose cord passes through the focus, $\phi+\chi=\frac{1}{2}\pi$, and h is the ordinate of the conjugate arc AD. Hence we derive this theorem,

Any focal arc of a parabola is equal to the difference between the conjugate apsidal arc and its ordinate.

The relation between the amplitudes ϕ and ω, in this case, is $\sin 2\phi=\dfrac{2\cos\omega}{1-\cos\omega}$. Thus when the focal cord makes an angle of 30° with the axis, we get $\cos\omega=\frac{1}{5}$, or $y=5k$. Here therefore the ordinate of the conjugate arc is five times the semiparameter.

We may in all cases represent by a simple geometrical construction the ordinates of the conjugate parabolic arcs, whose amplitudes are ϕ, χ, and ω.

Let ABC be a parabola whose focus is O, and whose vertex is A. Let $AO=g=\frac{1}{2}k$; moreover, let AB be the



arc whose amplitude is ϕ, and AC the arc whose amplitude is χ. At the points A, B, C draw tangents to the parabola, they will form a triangle circumscribing the parabola, whose sides represent the semi-ordinates of the conjugate arcs, AB, AC, AD.

We know that the circle circumscribing this triangle passes through the focus of the parabola. Now

$Ab = g\tan\phi$, $Ac = g\tan\chi$, $bd = g\tan\phi\sec\chi$, $cd = g\tan\chi\sec\phi$;

hence $\quad bd + cd = g(\tan\phi\sec\chi + \tan\chi\sec\phi)$,

therefore $\quad g\tan\omega = bd + cd.$

When AB, AC together constitute a focal arc, the angle adc is a right angle.

The diameter of this circle is $g\sec\phi\sec\chi$.

The demonstration of these properties follows obviously from the figure.

In the trigonometry of the circle, we find the formula

$$\vartheta = \tan\vartheta - \frac{\tan^3\vartheta}{3} + \frac{\tan^5\vartheta}{5} - \frac{\tan^7\vartheta}{7} + \&c. \ldots. \text{(a.)}$$

And if we develop by common division the expression

$$\frac{1}{\cos\theta} = \frac{\cos\theta}{1-\sin^2\theta} = \cos\theta(1 + \sin^2\theta + \sin^4\theta + \sin^6\theta + \ldots \&c.),$$

and integrate,

$$\int\frac{d\theta}{\cos\theta} = \sin\theta + \frac{\sin^3\theta}{3} + \frac{\sin^5\theta}{5} + \frac{\sin^7\theta}{7} + \&c.\ldots.\text{(b.)}$$

If we now inquire, what, in the circle, is the arc which differs from its protangent, by the distance between the vertex and its focus; or, as the protangent is 0 in the circle, and the focus is the centre; the question may be changed into this other, what is the trigonometrical tangent of the arc of a circle equal to the radius. This question is answered by putting 1 for ϑ in (a.), and reverting the series

$$1 = \tan(1) - \frac{\tan^3(1)}{3} + \frac{\tan^5(1)}{5} - \frac{\tan^7(1)}{7} + \&c.,$$

we should get, in functions of the numbers of Bernoulli, the value of $\tan(1)$, as is shewn in most treatises on trigonometry.

Let us now make a like inquiry in the case of the parabola, and ask what is the value of the amplitude which will give the difference between the arc of the parabola and its protangent, equal to the distance between the focus and the vertex of the parabola. Now if θ be this angle, we must have $(k.\theta) - g\sec\theta\tan\theta = g$. But, in general,

$$(k.\theta) - g\sec\theta\tan\theta = g\int\frac{d\theta}{\cos\theta},$$

Hence we must have, in this case, $\int\frac{d\theta}{\cos\theta}=1$. If we now revert the series (b), putting 1 for $\int\frac{d\theta}{\cos\theta}$, we shall get from this particular value of the series,

$$1=\sin\theta+\frac{\sin^3\theta}{3}+\frac{\sin^5\theta}{5}+\frac{\sin^7\theta}{7}+\&c.,$$

an arithmetical value for $\sin\theta$. This we shall find to be $\sin\theta=\frac{e^1-e^{-1}}{e^1+e^{-1}}$, e being the number called the base of the naperian logarithms. Hence $\sec\theta+\tan\theta=e$; or, if we write ε for this particular value of θ to distinguish it from every other,

$$\sec\varepsilon+\tan\varepsilon=e\ \ldots\ldots\ldots\ldots\ldots(345).$$

We are thus (for the first time it is believed) put in possession of the geometrical origin of that quantity so familiarly known to mathematicians, the naperian base. From the above equations we may derive

$$\sec\varepsilon=\frac{e^1+e^{-1}}{2},\quad \tan\varepsilon=\frac{e^1-e^{-1}}{2}\ \ldots\ldots\ldots\ldots(346),$$

$$\text{or } \tan\varepsilon=1{\cdot}175203015,\ \text{whence } \varepsilon={\cdot}8657606,$$
$$\text{or } \varepsilon=49^\circ\ 36'\ 15''.$$

The corresponding arc of the parabola will be found to be

$$(k.\theta)=k\left[1+\frac{2^1}{123}+\frac{2^3}{12345}+\frac{2^5}{1234567}\ \&c.\right]\ldots(347).$$

If we assume the theory of logarithms as known, we may at once arrive at this value, for in general

$$\int\frac{d\theta}{\cos\theta}=\log(\sec\theta+\tan\theta);$$

and as this is to be 1, we must have $\sec\theta+\tan\theta=e$, as before.

If we now extend this inquiry, and ask, what is the magnitude of the amplitude of the arc of the parabola which shall render the difference between the parabolic arc and its protangent equal to n times the distance between the focus and the vertex, we shall have, as before, by the terms of the question,

$$(k.\theta)-g\sec\theta\tan\theta=ng\ \ldots\ldots\ldots\ldots\ldots(348).$$

But, in general,

$$(k.\theta)-g\sec\theta\tan\theta=g\int\frac{d\theta}{\cos\theta};$$

hence we must have

$$n = \int \frac{d\theta}{\cos\theta} = \log(\sec\theta + \tan\theta), \text{ or } \sec\theta + \tan\theta = e^n \ldots (349).$$

Now we may solve this equation in two ways, either by making n a given number, and then determine the value of $\sec\theta + \tan\theta$, which may be called the *base*. Or we may assign an arbitrary value to $\sec\theta + \tan\theta$, and then derive the value of n. Taking the latter course, let, for example,

$$\sec\theta + \tan\theta = 10, \quad \text{then } n = \log 10,$$

or $\frac{1}{n}$ is the modulus of the second system of logarithms. Hence, if we assume any number of systems of logarithms on the same parabola, and take their bases

$$(\sec\theta + \tan\theta), \quad (\sec\theta' + \tan\theta'), \quad (\sec\theta'' + \tan\theta''), \ldots \&c.,$$

the moduli of these successive systems will be the ratios of half the semiparameter to the successive differences between the base parabolic arcs and their protangents.

In the naperian system, g the distance from the focus to the vertex of the parabola, is taken as 1. The difference between the parabolic arc and its protangent when equal to g, gives $g(\sec\theta + \tan\theta) = eg$. In the decimal system $g(\sec\theta_{,} + \tan\theta_{,}) = 10g$, and the difference between the corresponding parabolic arc and its protangent being ng, if we make this difference ng equal to the arithmetical unit, we shall have $ng = 1$, or $g = \frac{1}{n} =$ modulus of the system. Hence, in every system of logarithms whatever, g the distance between the focus and the vertex of the parabola, is the modulus of the system. Every system of logarithms may be derived from the same parabola, but the naperian system, in which the focal distance of the vertex is itself taken as the unit, may justly be taken as the *natural* system. In the same way we may consider that to be the *natural* system of circular trigonometry, in which the radius is taken as the unit. The modulus in the trigonometry of the parabola corresponds with the radius in the trigonometry of the circle. But while the base in the trigonometry of the parabola is real, in the circle it is imaginary. In the parabola, the angle of the base is given by the equation $\sec\theta + \tan\theta = e$. In the circle, $\cos\theta + \sqrt{(-1)}\sin\theta = e^{\theta\sqrt{(-1)}}$, and making $\vartheta = 1$, we get

$$\cos(1) + \sqrt{(-1)}\sin(1) = e^{\sqrt{(-1)}}.$$

Hence, while e^1 is the *parabolic* base, $e^{\sqrt{(-1)}}$ is the *circular* base. Or as $[\sec \varepsilon + \tan \varepsilon]$ is the naperian base, $[\cos(1) + \sqrt{(-1)}\sin(1)]$ is the *circular* or imaginary base. Thus

$$[\cos(1) + \sqrt{(-1)}\sin(1)]^{\vartheta} = \cos\vartheta + \sqrt{(-1)}\sin\vartheta.$$

Hence, speaking more precisely, imaginary numbers have real logarithms, but an imaginary base. We may always pass from the real logarithms of the parabola, to the imaginary logarithms of the circle, by changing $\tan\theta$ into $\sqrt{(-1)}\sin\vartheta$, $\sec\theta$ into $\cos\vartheta$, and e^1 into $e^{\sqrt{(-1)}}$.

As in the parabola, the angle θ is non-periodic, its limit being $\frac{1}{2}\pi$, while in the circle ϑ has no limit, it follows that while a number can have only one real or *parabolic* logarithm, it may have innumerable imaginary or *circular* logarithms.

In the parabola we thus can shew the geometrical origin of the magnitudes known as the base and the modulus. We might too form systems of circular trigonometry analogous to different systems of logarithms. We might refer the arc of a circle not to the radius, but to some other arbitrary fixed line, the diameter or any other suppose. Let the circumference be referred to the diameter, then π will signify a whole circumference instead of a semicircle, and $\frac{1}{4}\pi$ will represent a right angle. Having on this system, or any similar one, found the lengths of the arcs which correspond to certain functions, such as given sines or tangents, we should multiply the results by some fixed number, which we might call a modulus (2 in this example), to reduce them to the standard system; but such systems would obviously be useless.

If ε be the angle which gives the difference between the parabolic arc and its protangent equal to $g = \frac{1}{2}k$; $(\varepsilon \perp \varepsilon)$ is the angle which will give this difference equal to $2g$, $(\varepsilon \perp \varepsilon \perp \varepsilon)$ is the angle which will give this difference equal to $3g$, and so on to any number of angles. Hence, in the circle, if ϑ be the angle which gives the circular arc equal to the radius, 2ϑ is the angle which will give an arc equal to twice the radius, and so on for any number of angles. This is of course self-evident in the case of the circle, but it is instructive to point out the complete analogy which holds in the trigonometries of the circle and of the parabola.

The geometrical origin of the exponential theorem may thus be shewn.

Assume two known logarithmic bases $(\sec\alpha + \tan\alpha)$, and $(\sec\beta + \tan\beta)$, and let us investigate the ratio of the differences of the corresponding parabolic arcs and their protangents.

Let $\sec\varepsilon + \tan\varepsilon$ be the naperian base, and let one difference be xg and the other yg. The ratio of these differences is therefore $\frac{y}{x} = z$, if we make $y = xz$. Hence

$$\sec\alpha + \tan\alpha = (\sec\varepsilon + \tan\varepsilon)^x = e^x, \text{ and } (\sec\beta + \tan\beta) = e^y.$$

Therefore $(\sec\alpha + \tan\alpha)^y = e^{xy} = (\sec\beta + \tan\beta)^x$.

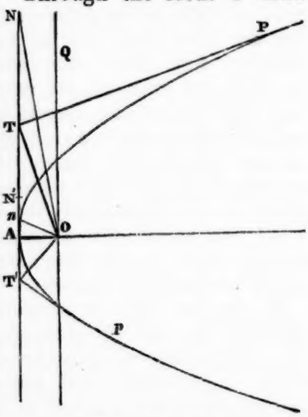
Or, as $y = xz$, $(\sec\alpha + \tan\alpha)^z = \sec\beta + \tan\beta$.

Let A be the first base, and B the second. Then $B = A^z$. This is the exponential theorem.

Let A be the naperian base, then $x = 1$, and $A = e$. Hence $B = e^y$.

Given a number to find its logarithm, may be exhibited by the following geometrical construction.

Let OAP be a parabola. Through the focus O draw the perpendicular OQ to the axis AO. Through A let a tangent of indefinite length be drawn. On this tangent take the line AN to represent the given number. Join NO, and make the angle NOT always equal to the angle NOQ. Draw TP at right angles to TO. This line will touch the parabola in the point P, and the arc of the parabola $AP - PT$ will be the logarithm of AN.



When $AN' = AO =$ the unit g, the angle $N'OQ$ is equal to half a right angle. Hence the point T in this case will coincide with A. The parabolic arc therefore vanishes, or the logarithm of 1 is 0. When $\sec\theta + \tan\theta = 1$, $\theta = 0$.

When the number is less than 1, the point N will fall below N' in the position n. Hence nOQ is greater than half a right angle. Therefore T will fall *below* the axis in the point T'; and if we draw through T' a tangent $T'p$, it will give the *negative* arc of the parabola $T'p$, corresponding to the number An. Fractional numbers, or numbers between $+1$ and 0, must therefore be represented by the expression $g(\sec\theta - \tan\theta)$, since $\tan\theta$ changes its sign.

When the number is 0, n coincides with A, and the angle NOQ in this case is a right angle. Therefore the point T' will be the intersection of AT' and OQ. Hence T' is at an infinite distance below the axis, and therefore the logarithm of $+0$ is $-\infty$.

Hence negative numbers have no logarithms, at least no real ones; and imaginary ones can only be educed by the transformation so often referred to, and this leads us to seek them among the properties of the circle. For as θ always lies between 0 and a right angle, or between 0 and the half of $\pm\pi$, $\sec\theta \pm \tan\theta$ is *always* positive; hence *negative* numbers can have no real or *parabolic* logarithms, but they may have imaginary or *circular* logarithms; for in the expression $\log\{\cos\vartheta + \sqrt{(-1)}\sin\vartheta\} = \vartheta\sqrt{(-1)}$, we may make $\vartheta = (2n+1)\pi$, and we shall get $\log(-1) = (2n+1)\pi\sqrt{(-1)}$.

Hence also, as the length of the parabolic arc TP, without reference to the sign, depends solely on the amplitude θ, it follows that the logarithm of $\sec\theta - \tan\theta$ is equal to the logarithm of $\sec\theta + \tan\theta$. As $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$, we may hence infer, that the logarithm of any number is equal to the logarithm of its reciprocal, with the sign changed.

When θ is very large, $\sec\theta + \tan\theta = 2\tan\theta$, nearly. Hence if we represent a large number by an ordinate of a parabola whose focal distance to the vertex is 1, the difference between the corresponding arc and its protangent will represent its logarithm.

Along the tangent to the vertex of the parabola, as in the preceding figure, draw, measured from the vertex, a series of lines in geometrical progression,

$$g(\sec\theta + \tan\theta),\ g(\sec\theta + \tan\theta)^2,$$
$$g(\sec\theta + \tan\theta)^3 \ldots\ldots g(\sec\theta + \tan\theta)^n.$$

Join N, the general representative of the extremities of these right lines, with the focus O. Erect the perpendicular OQ, and make the angle NOT *always* equal to the angle NOQ. The line OT will be $= g\sec\theta$, the line $OT_{\prime} = g\sec(\theta \perp \theta)$, the line $OT_{\prime\prime} = g\sec(\theta \perp \theta \perp \theta)$, &c., and we shall likewise have

$$AT = g\tan\theta,\ AT_{\prime} = g\tan(\theta \perp \theta),\ AT_{\prime\prime} = g\tan(\theta \perp \theta \perp \theta),\ \&c.$$

This follows immediately from (342); for any integral power of $(\sec\theta + \tan\theta)$ may be exhibited as a linear function of $\sec\Theta + \tan\Theta$. If $\Theta = \theta \perp \theta \perp \theta \ldots$ &c., since

$$\sec(\theta \perp \theta \perp \theta \perp \theta \text{ \&c. to } n\theta) + \tan(\theta \perp \theta \perp \theta \perp \theta \text{ \&c. to } n\theta) = (\sec\theta + \tan\theta)^n.$$

Hence the parabola enables us to give a graphical construction for the angle $(\theta \perp \theta \perp \&c.)$ as the circle does for the angle $(\theta + \theta + \&c.)$.

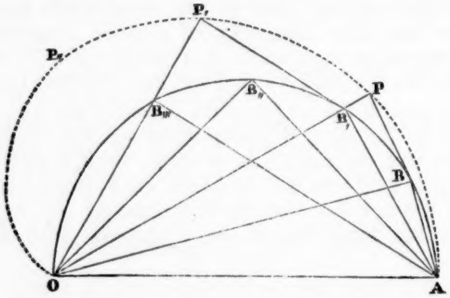
The analogous theorem in the circle may be developed as follows:—In the circle OBA take the arcs

$$AB = BB_{,} = B_{,}B_{,,} = B_{,,}B_{,,,} \ldots \&c. = 2\vartheta.$$

Let the diameter be G; then

$$OB = G\cos\vartheta,\ OB_{,} = G\cos 2\vartheta,\ OB_{,,} = G\cos 3\vartheta \ldots \&c.$$

and $AB = G\sin\vartheta,\ AB_{,} = G\sin 2\vartheta,\ AB_{,,} = G\sin 3\vartheta \ldots \&c.$



Now as the lines in the second group are always at right angles to those in the first, and as such a change is denoted by the symbol $\sqrt{(-1)}$, we get

$$OB + BA = G\{\cos\vartheta + \sqrt{(-1)}\sin\vartheta\},$$

$$OB_{,} + B_{,}A = G\{\cos 2\vartheta + \sqrt{(-1)}\sin 2\vartheta\} = G\{\cos\vartheta + \sqrt{(-1)}\sin\vartheta\}^2;$$

$$OB_{,,} + B_{,,}A = G\{\cos 3\vartheta + \sqrt{(-1)}\sin 3\vartheta\} = G\{\cos\vartheta + \sqrt{(-1)}\sin\vartheta\}^3 \ \&c.$$

The known theorem, that a parabola is the reciprocal polar of a circle, whose circumference passes through the focus, suggests a transformation, which will exhibit a much closer analogy between the formulæ for the rectification of the parabola and the circle, than when the centre of the latter curve is taken as the origin.

Let OBA be a semicircle; let the origin be placed at O; let the angle $AOB = \vartheta$; and let G, as before, be the diameter of the circle. Through B draw the tangent BP; let fall on this tangent the perpendicular $OP = p$, and let BP, the protangent, be equal to t.

Now as $p = G\cos^2\vartheta$, and $t = G\sin\vartheta\cos\vartheta$, as also the angle $AOP = 2\vartheta$, if we apply to the circle the formula for rectifica-

tion in (33), we shall have the arc

$$AB = s = 2G\int \cos^2\vartheta d\vartheta - G\sin\vartheta\cos\vartheta.$$

Make the imaginary transformations $\cos\vartheta = \sec\theta$, and $\sin\vartheta = \sqrt{(-1)}\tan\theta$, and we shall have

$$\frac{s}{G\sqrt{(-1)}} = 2\int\frac{d\theta}{\cos^3\theta} - \sec\theta\tan\theta,$$

the expression for an arc of a parabola, diminished by its protangent.

The protangent to the circle, which is exhibited in this formula, disappears in the actual process of integration; while in the parabola, the protangent which is involved in the differential, is evolved by the process of integration.

As in the parabola, the perpendicular, from the focus on the tangent, bisects the angle between the radius vector and the axis of the curve; so in the circle, the radius vector OB drawn from the extremity of the diameter, bisects the angle between the perpendicular OP and the diameter OA.

There are some curious analogies between the parabola and the circle, considered under this point of view.

In the parabola, the points T, $T_{,}$, $T_{,,}$, which divide the lines

$$g(\sec\theta + \tan\theta), \quad g[\sec(\theta \dotplus \theta) + \tan(\theta \dotplus \theta)]$$

into their component parts, are upon tangents to the parabola. The corresponding points B, $B_{,}$, $B_{,,}$ in the circle, are on the circumference of the circle.

In the parabola the extremities of the lines $g(\sec\theta + \tan\theta)$ are on a right line AT; in the circle the extremities of the bent lines are all in the point A.

The locus of the point T, the intersections of the tangents to the parabola with the perpendiculars from the focus, is a right line; or in other words, while one end of a protangent rests on the parabola, the other end rests on a right line. So in the circle, while one end of the protangent rests on the circle, the other end rests on a *cardioide*, whose diameter is equal to that of the circle, and whose cusp is at O. $OPP_{,}A$ is the cardioide.

The length of the tangent AN to any point N is

$$g(\sec\theta + \tan\theta) = 2g\tan\theta,$$

when θ is very large. The length of the cardioide is $2G\sin\vartheta$.

It is singular that the imaginary formulæ in trigonometry have long been discovered, while the corresponding real expressions have escaped notice. Indeed, it was long ago

observed by Bernoulli, Lambert, and by others—the remark has been repeated in almost every treatise on the subject since—that the ordinates of an equilateral hyperbola might be expressed by real exponentials, whose exponents are sectors of the hyperbola, but the analogy, being illusory, never led to any useful results. And the analogy was illusory from this, that it so *happens* the length and area of a circle are expressed by the *same* function, while the area of an equilateral hyperbola is a function of an arc of a parabola. The true analogue of the circle is the parabola.

Let $\bar{\omega}$ be the conjugate amplitude of ω and ψ, while ω is the conjugate amplitude, as before, of ϕ and χ. Then, as

$$\int\frac{d\bar{\omega}}{\cos\bar{\omega}} = \int\frac{d\omega}{\cos\omega} + \int\frac{d\psi}{\cos\psi}, \text{ and } \int\frac{d\omega}{\cos\omega} = \int\frac{d\phi}{\cos\phi} + \int\frac{d\chi}{\cos\chi},$$

we shall have

$$\int\frac{d\bar{\omega}}{\cos\bar{\omega}} = \int\frac{d\phi}{\cos\phi} + \int\frac{d\chi}{\cos\chi} + \int\frac{d\psi}{\cos\psi};$$

and if $(k.\bar{\omega})$, $(k.\phi)$, $(k.\chi)$, and $(k.\psi)$ are four corresponding parabolic arcs,

$$(k.\bar{\omega})-(k.\phi)-(k.\chi)-(k.\psi)=k\tan(\phi\perp\chi)\tan(\phi\perp\psi)\tan(\chi\perp\psi)\ldots(350),$$

which gives a simple relation between four conjugate parabolic arcs.

Let, in the preceding formula, $\phi=\chi=\psi$, and we shall have

$$(k.\bar{\omega}) - 3\,(k.\phi) = k\tan^3(\phi \perp \phi) = 8k\tan^3\phi\sec^3\phi\ldots(351).$$

We are thus enabled to assign the difference between an arc of a parabola and three times another arc, $\bar{\omega} = (\phi \perp \phi \perp \phi)$.

If in (ϖ), (341), we make $\phi = \chi = \psi$, $\tan\bar{\omega} = 4\tan^3\phi + \tan\phi$.

Introduce into this expression the imaginary transformation $\tan\phi = \sqrt{(-1)}\,\sin\theta$, and we shall get $\sin 3\theta = -4\,\sin^3\theta + \sin\theta$, which is the known formula for the trisection of a circular arc. (351) may therefore be taken as the formula which gives the trisection of an arc of a parabola.

When there are five parabolic arcs, whose normal angles ϕ, χ, ψ, v, Ω are related as above, namely,

$$\omega = \phi \perp \chi, \quad \bar{\omega} = \omega \perp \psi = \phi \perp \chi \perp \psi, \quad \Omega = \phi \perp \chi \perp \psi \perp v,$$

we get the following relation,

$$(k.\Omega) - (k.\phi) - (k.\chi) - (k.\psi) - (k.v)$$
$$= k\tan(\phi \perp \chi \perp v)\tan(\chi \perp \psi \perp v)\tan(\psi \perp \phi \perp v)\ldots(352),$$

a formula which connects five parabolic arcs, whose amplitudes are derived by the given law.

The theorem given in (342) is a particular case of this more general theorem

$$\sec(\alpha \dotplus \beta \dotplus \gamma \dotplus \delta \dotplus \&c.) + \tan(\alpha \dotplus \beta \dotplus \gamma \dotplus \delta \dotplus \&c.)$$
$$= (\sec\alpha + \tan\alpha)(\sec\beta + \tan\beta)(\sec\gamma + \tan\gamma)(\sec\delta + \tan\delta)\ \&c.$$

We might pursue this subject very much further; but enough has been done to show the analogy which exists between the trigonometry of the circle and that of the parabola. As the calculus of angular magnitude has always been referred to the circle as its type, so the calculus of logarithms may, in precisely the same way, be referred to the parabola as its type.

The obscurities, which hitherto have hung over the geometrical theory of logarithms, have it is hoped been now removed. It is possible to represent logarithms, as elliptic integrals usually have been represented, by curves devised to exhibit some special property only; and accordingly, such curves, while they place before us the properties they have been devised to represent, fail generally to carry us any further. The close analogies which connect the theory of logarithms with the properties of the circle will no longer appear inexplicable*.

ON ELECTRODYNAMIC INDUCTION.

By Riccardo Felici.

[*Extracted from a Letter to the Editor.*]

* * * Monsieur Tortolini m'a écrit que vous voudriez bien insérer, par extrait, mes travaux dans votre Journal accrédité. Je vous remercie infiniment de l'offre, de laquelle je profite dés ce moment; en vous priant d'accueillir l'extrait sui-

* The views above developed, on the trigonometry of the parabola, throw much light on a controversy long carried on between Leibnitz and J. Bernoulli on the subject of the logarithms of negative numbers. Leibnitz insisted they were imaginary, while Bernoulli argued they were real, and the same as the logarithms of equal positive numbers. Euler espoused the side of the former, while D'Alembert coincided with the views of Bernoulli. Indeed, if we derive the theory of logarithms from the properties of the hyperbola (as geometers always have done), it will not be easy satisfactorily to answer the argument of Bernoulli—that as an hyperbolic area represents the logarithm of a positive number, denoted by the positive abscissa $+x$, so a negative number, according to conventional usage, being represented by the negative abscissa $-x$, the corresponding hyperbolic area should denote its logarithm also. All this obscurity is cleared up by the theory developed in the text, which completely establishes the correctness of the views of Leibnitz and Euler.

vant, d'un mémoire qui sera publié prochainement dans les *Annales de l'Université de Toscane.*

Mémoire sur l'Induction Electro-Dynamique. (*Extrait*).

A' l'aide d'un nouveau méthode expérimental que l'on trouverá decrit dans les *Annales des Sciences,* publiés á Rome, par M. le Professeur Tortolini, année 1851, il est facile d' établir avec toute certitude les faits suivants.

1. La force des courants induits en ouvrant, ou bien en fermant, le circuit de la pile, est simplement proportionelle à celle des courants inducteurs.

2. Le théorème relatif au conducteur sinueux, énoncé dans la théorie d'Ampère, se vérifie aussi dans le cas de l'induction.

3. Dans le cas de deux anneaux, dont l'un est l'induit et l'autre l'inducteur, égaux, paralléles et avec leurs centres sur la même droite normale à leurs plans, la force des courants induits, en interrompant le circuit de la pile, croit proportionellement aux diamétres, lorsque le rapport qui existe entre les distances des ces plans et les diamétres des mêmes anneaux, est une quantité constante.

4. La somme A de tous les courants induits sur un circuit conducteur par un circuit voltaïque, fermé et en mouvement, pendant que ce dernier circuit passe d'une position (dans laquelle il ne pourrait produire soit en l'ouvrant soit en le fermant aucun courant induit sur le prémier conducteur) à une autre position quelconque, est égale au courant B induit que l'on peut obtenir en ouvrant ou en fermant le même circuit inducteur placé exactement dans la derniere position. Il est évident que c'est du mouvement relatif, des deux circuits, que l'on entend parler.

Celà posé, et en suivant la mêthode suivie par Ampère, dans la théorie des phénomènes électro-dynamiques, on voit que les faits 1e, 2e, déterminent la forme la plus générale possible de la fonction algébrique qui exprime la force électro-motrice, d^2E, induite sur l'élément ds dans la direction du même élément, par un élément ds' inducteur, en ouvrant ou bien en fermant le circuit de la pile. Et par consequent on aura la formule

$$d^2E = -\,ar^{1-n-k}d\,\frac{\left(r^k\dfrac{dr}{ds}\right)}{ds'}.ds.ds^2 \ldots\ldots\ldots (1),$$

où r dénote la distance des éléments; a une quantité proportionelle à la force de la pile ; k, n constantes qui doivent être déterminées par l'expérience. Mais la valeur de n est facile a connaître en vertu du 3[e] fait; et par un calcul très-facile on trouve $n = 1$; c'est-à-dire la formule

$$d^2E = -a\left(\frac{d^2r}{ds.ds'} + \frac{k}{r}\frac{dr}{ds}\cdot\frac{dr}{ds'}\right)ds.ds^2 \ldots\ldots (2).$$

On peut maintenant remarquer, que le premier terme de son second membre, disparait dans les intégrations, pour un circuit fermé, et l'on ecrira

$$d^2E = -\frac{ak}{r}\cdot\frac{dr}{ds}\cdot\frac{dr}{ds'}.ds.ds^2 \ldots\ldots\ldots\ldots (3).$$

Pour le cas des courants induits par le mouvement du circuit inducteur, on voit, très-clairement que le fait 4[e] resout le problême sans ajouter à l'analyse aucune nouvelle difficulté.

La formule (3) donne des resultats assez simples lorsque on l'applique au cas du magnétisme dans l'hypothése de Ampère.

Voilà, Monsieur, ce que j'avais a vous communiquer de plus important dans le dit mémoire. * * *

Pisa, March 30, 1852.

ON THE INDEX SYMBOL OF HOMOGENEOUS FUNCTIONS.

By ROBERT CARMICHAEL, A.M., Fellow of Trinity College, Dublin.

[*Concluded from* Vol. VII. p. 284.]

IN the first article of the following paper, the Index Symbol is employed to illustrate a useful general theorem in the Calculus of Operations. In the second and third articles, the general applicability of the same symbol to the integration of large classes of differential equations, ordinary and partial, is exhibited and illustrated. In the fourth article, this symbol is employed for the solution of systems of simultaneous partial differential equations, a department of the Integral Calculus as yet comparatively unnoticed, and possessing much that will interest the mathematician and physicist. Finally, in the fifth article, certain excep-

tional cases are discussed, which will occur in this, as in any other, method of integration.

1. ψ being a distributive symbol such that

$$\psi.uv = u\psi v + v\psi u,$$

it can be readily proved that

$$e^{\psi}.uv = e^{\psi}u.e^{\psi}v,$$

whence it follows that

$$e^{\psi}.uvw\ldots = e^{\psi}u.e^{\psi}v.e^{\psi}w\ldots$$

Hence $$e^{\psi}.u^n = (e^{\psi}u)^n,$$

and therefore, if F denote any algebraic function,

$$e^{\psi}.F(u) = F(e^{\psi}u).$$

This valuable theorem is due to the Rev. Prof. Graves.

Now the distributive symbol

$$x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz} + \&\text{c.} = \nabla,$$

satisfies the above law, and therefore

$$e^{\Theta\nabla}.F(U) = F(e^{\Theta\nabla}U) \ldots\ldots\ldots\ldots\ldots (\text{I.}),$$

where Θ and U are any functions whatsoever of xyz &c.

With this symbol ∇, however, are connected* the two useful theorems

$$F(\nabla).u_m = F(m).u_m, \quad F(\nabla).u_m V = u_m.F(\nabla + m)V\ldots(\text{II.}),$$

where u_m is an homogeneous function of the m^{th} degree in xyz &c., the generalizations of the elementary formulæ

$$F\left(x\frac{d}{dx}\right)x^m = F(m)\,x^m, \quad F\left(x\frac{d}{dx}\right)x^m X = x^m F\left(x\frac{d}{dx} + m\right)X,$$

better known in the shape

$$F\left(\frac{d}{d\theta}\right)e^{m\theta} = F(m)\,e^{m\theta}, \quad F\left(\frac{d}{d\theta}\right)e^{m\theta}\Theta = e^{m\theta}F\left(\frac{d}{d\theta} + m\right)\Theta.$$

Hence, if U be a mixed rational function of xyz &c., broken up into its homogeneous terms, thus

$$U = u_0 + u_1 + u_2 + \&\text{c.} + u_n,$$

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* *Camb. and Dub. Math. Journal*, Nov. 1851. *Phil. Mag.*, Feb. 1852.

we obtain the remarkable theorem

$$e^{\nabla}.F(U) = F(u_0 + eu_1 + e^2u_2 + \&c. + e^nu_n).$$

As a second example of the general theorem (I.) we may investigate the algebraic value of the symbolic quantity

$$e^{\Theta_m\nabla}.F(\Theta_n),$$

where Θ_m, Θ_n are known homogeneous functions of the degrees *m, n,* respectively. Now

$$e^{\Theta_m\nabla}.\Theta_n = \left\{1 + n\Theta_m + \frac{n(n+m)}{1.2}\Theta^2_m + \&c.\right\}\Theta_n = \frac{\Theta_n}{(1-m\Theta_m)^{\frac{n}{m}}},$$

and therefore $\quad e^{\Theta_m\nabla}.F(\Theta_n) = F\left\{\dfrac{\Theta_n}{(1-m\Theta_m)^{\frac{n}{m}}}\right\}.$

Thus $\quad e^{(ax+by+cz)\nabla}.F(x^2+y^2+z^2) = F\left[\dfrac{x^2+y^2+z^2}{\{1-(ax+by+cx)\}^2}\right].$

2. There is one class of linear differential equations with *constant coefficients,* whose integration, in general, presents insurmountable difficulties to the student. In it the right-hand member or absolute term consists exclusively of exponentials or circular functions, sines, cosines, &c., and may be written in the form

$$f(e^\theta, \sin\theta, \cos\theta).$$

For the solution of such equations, different processes have been employed, varying with the character of each example, useless in practice when the order of the equation is elevated, and unsuggestive of any susceptibilities of extended application.

It is proposed to shew that, through the instrumentality of the Index Symbol, this class of equations can be integrated by a process simple and uniform, equally susceptible of employment in equations of the higher orders as in those of the lower, and directly indicative, in either case, of a corresponding class of partial differential equations with the appropriate form of solution.

The type of the first class is

$$\frac{d^my}{d\theta^m} + P\frac{d^{m-1}y}{d\theta^{m-1}} + Q\frac{d^{m-2}y}{d\theta^{m-2}} + \ldots + Ty = f(e^\theta, \sin\theta, \cos\theta).$$

This, being reduced to the form

$$F\left(\frac{d}{d\theta}\right)y = \Sigma Ae^{n\theta},$$

where a may be positive or negative, fractional or integer, real or imaginary, becomes, by the substitution $x = e^\theta$,

$$F\left(x\frac{d}{dx}\right)y = \Sigma A x^a,$$

and the solution, obtained at once in terms of x, gives in terms of θ

$$y = \Sigma A \frac{e^{a\theta}}{F(a)} + \textit{ord. comp. funct.}$$

and when a is imaginary, we may restore the circular function.

We have said that this class of ordinary differential equations has its analogue amongst partial differential equations, and that the method of solution of the former is directly suggestive of that of the latter.

Thus, the class of partial differential equations whose type is

$$\nabla^m z + P\nabla^{m-1}z + \ldots + Tz = f(e^\theta, e^\phi, \sin\theta, \sin\phi, \cos\theta, \cos\phi),$$

where
$$\nabla = \frac{d}{d\theta} + \frac{d}{d\phi}$$

can be thrown into the form

$$F(\nabla)\,z = \Sigma A_{a,b}\, e^{a\theta + b\phi};$$

where, as before, a and b may be positive or negative, integer or fractional, real or imaginary. By the transformations $x = e^\theta$, $y = e^\phi$, this becomes

$$F\left(x\frac{d}{dx} + y\frac{d}{dy}\right)z = F(\nabla)\,z = \Sigma A_{a,b}\, x^a y^b,$$

and the solution, obtained at once in terms of x, y by the method furnished in either of the papers before quoted, is in terms of θ, ϕ,

$$z = \Sigma A_{a,b} \frac{e^{a\theta + b\phi}}{F(a+b)} + \textit{corresponding comp. funct.}$$

When the roots of $F(\nabla) = 0$ are all real and unequal, this complementary function or arbitrary portion of the solution is of the form

$$\psi_n(e^\theta, e^\phi) + \psi_p(e^\theta, e^\phi) + \psi_q(e^\theta, e^\phi) + \text{\&c.},$$

where n, p, q, &c. are the values of the roots, and ψ_n, ψ_p, ψ_q, &c. are homogeneous functions of the *given degrees* n, p, q, *&c. but whose forms are arbitrary.*

When there are α equal roots whose common value is n, its form is

$$\psi_n(e^\theta, e^\phi).(\theta+\phi)^{\alpha-1} + \chi_n(e^\theta, e^\phi).(\theta+\phi)^{\alpha-2} + \&c. + \psi_p(e^\theta, e^\phi) + \&c.,$$

where ψ_n, χ_n, &c. are different arbitrary homogeneous functions of the degree n.

Finally, when there are pairs of imaginary roots, the form of the arbitrary portion of the solution is

$$\psi_{n+p\sqrt{(-1)}}(e^\theta, e^\phi) + \psi_{n-p\sqrt{(-1)}}(e^\theta, e^\phi) + \&c. + \psi_q(e^\theta, e^\phi) + \&c.$$

We proceed to furnish some illustrations of the above method of solution, which would seem to establish its value as a practical good. The equations proposed for solution are selected from *Gregory's Examples.*

$$\text{(I.)} \qquad \frac{d^2y}{d\theta^2} - 2m\frac{dy}{d\theta} + m^2y = \sin a\theta.$$

The method of investigation given for the solution of this simple equation is extremely artificial, and seemingly unsusceptible of extension.

By the transformation $x = e^\theta$, it becomes

$$\left(x\frac{d}{dx} - m\right)^2 y = \frac{1}{2\sqrt{(-1)}}\left\{x^{a\sqrt{(-1)}} - x^{-a\sqrt{(-1)}}\right\};$$

therefore

$$y = \frac{1}{2\sqrt{(-1)}}\left[\frac{x^{a\sqrt{(-1)}}}{\{a\sqrt{(-1)}-m\}^2} - \frac{x^{-a\sqrt{(-1)}}}{\{a\sqrt{(-1)}+m\}^2}\right] + c_1x^m\log x + c_2x^m;$$

or, replacing the circular function,

$$y = \frac{(m^2-a^2)\sin a\theta + 2ma\cos a\theta}{(m^2+a^2)^2} + e^{m\theta}(c_1\theta + c_2).$$

By the transformations $x = e^\phi$, $y = e^\psi$, and a repetition of the same precise process which we have now employed, can be obtained the solution of the partial differential equation

$$\left(\frac{d^2z}{d\phi^2} + 2\frac{d^2z}{d\phi\, d\psi} + \frac{d^2z}{d\psi^2}\right) - 2m\left(\frac{dz}{d\phi} + \frac{dz}{d\psi}\right) + m^2z = \sin(a\theta + b\phi),$$

in the form

$$z = \begin{cases} \dfrac{\{m^2-(a+b)^2\}\sin(a\theta+b\phi) + 2m(a+b)\cos(a\theta+b\phi)}{\{m^2+(a+b)^2\}^2} \\ \qquad + \Phi_m(e^\phi, e^\psi).(\phi+\psi), \quad + \Psi_m(e^\phi, e^\psi), \end{cases}$$

where Φ_m, Ψ_m are different arbitrary homogeneous functions of the m^{th} degree.

(II.) $$x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}.$$

Expanding the right-hand member, this becomes

$$\left(x\frac{d}{dx} + 1\right)^2 y = 1 + 2x + 3x^2 + \&c.;$$

therefore $$y = \left(1 + \frac{x}{2} + \frac{x^2}{3} + \&c.\right) + \frac{c_1}{x}\log x + \frac{c_2}{x},$$

or $$y = \log\left(\frac{1}{1-x}\right)^{\frac{1}{x}} + \frac{c_1}{x}\log x + \frac{c_2}{x}.$$

Similarly, the solution of the partial differential equation

$$(x^2r + 2xys + y^2t) + 3(xp + yq) + z = \frac{1}{(1-\Theta_1)^2},$$

where Θ_1 is a given homogeneous function of the first degree in x, y, is, by the second fundamental theorem,

$$z = \log\left(\frac{1}{1-\Theta_1}\right)^{\frac{1}{\Theta_1}} + u_{-1}(\log x + \log y) + v_{-1},$$

where u_{-1}, v_{-1} are different arbitrary homogeneous functions in x, y, of the degree -1.

(III.) $$x\frac{dw}{dx} + y\frac{dw}{dy} + z\frac{dw}{dz} - aw = \frac{xy}{z}.$$

Thrown into the symbolic shape, this becomes

$$(\nabla - a)\, w = \frac{xy}{z},$$

and therefore the solution is

$$w = \frac{1}{1-a}\,\frac{xy}{z} + u_a,$$

where u_a is an arbitrary homogeneous function in the quantities x, y, z, of the degree a.

More generally, the solution of

$$x\frac{dw}{dx} + y\frac{dw}{dy} + z\frac{dw}{dz} - aw = \frac{\Theta_m}{\Theta_n},$$

where Θ_m, Θ_n are known homogeneous functions of x, y, z, of the m^{th} and n^{th} degrees respectively, is

$$w = \frac{1}{(m-n)-a}\,\frac{\Theta_m}{\Theta_n} + u_a.$$

$$\text{(IV.)}\quad x^n \frac{d^n z}{dx^n} + nx^{n-1} y \frac{d^n z}{dx^{n-1}dy} + \frac{n(n-1)}{1.2} x^{n-2} y^2 \frac{d^n z}{dx^{n-2}dy^2} + \ldots = 0.$$

By the third article of the paper, to which the writer has already taken the liberty of referring, it appears most readily that this equation is susceptible of the symbolic shape

$$\nabla(\nabla - 1)(\nabla - 2)\ldots(\nabla - n + 1)z = 0.$$

Consequently its solution is, at once,

$$z = u_0 + u_1 + u_2 + \ldots + u_{n-1}.$$

More generally, the solution of the equation

$$x^n \frac{d^n z}{dx^n} + nx^{n-1} y \frac{d^n z}{dx^{n-1}dy} + \frac{n(n-1)}{1.2} x^{n-2} y^2 \frac{d^n z}{dx^{n-2}dy^2} + \ldots = \Theta_a + \Theta_b,$$

is

$$z = \frac{\Theta_a}{a(a-1)\ldots(a-n+1)} + \frac{\Theta_b}{b(b-1)\ldots(b-n+1)} + u_0 + u_1 + \ldots + u_{n-1}.$$

The simplicity of the method employed in this last example, when compared with the artificial and laborious processes which have hitherto been employed for its solution, seems to exhibit, in a remarkable degree, the power of the Index Symbol as an instrument of integration, and the facility with which it admits of manipulation. It is obvious that (11) in the Examples is only a particular case of the general theorem now established.

3. Hitherto we have confined our attention to the integration of classes of differential equations, ordinary and partial, in which the coefficients are constants. There are, however, two classes of equations in which the coefficients are symmetric functions of the variables, which can easily be reduced to those discussed.

In the first, the coefficients are symmetric functions of the independent variables. Its type is

$$\left.\begin{array}{l} A\left\{(m+\lambda x)^\alpha \dfrac{d^\alpha z}{dx^\alpha} + \alpha(m+\lambda x)^{\alpha-1}(n+\lambda y)\dfrac{d^\alpha z}{dx^{\alpha-1}dy} \right. \\ \qquad \left. + \dfrac{\alpha(\alpha-1)}{1.2}(m+\lambda x)^{\alpha-2}(n+\lambda y)^2 \dfrac{d^\alpha z}{dx^{\alpha-2}dy^2} + \&\text{c.}\right\} \\ \qquad\qquad + \\ B\left\{(m+\lambda x)^\beta \dfrac{d^\beta z}{dx^\beta} + \beta(m+\lambda x)^{\beta-1}(n+\lambda y)\dfrac{d^\beta z}{dx^{\beta-1}dy} \right. \\ \qquad \left. + \dfrac{\beta(\beta-1)}{1.2}(m+\lambda x)^{\beta-2}(n+\lambda y)^2 \dfrac{d^\beta z}{dx^{\beta-2}dy^2} + \&\text{c.}\right\} \\ \qquad\qquad + \&\text{c.} \end{array}\right\} = \Omega.$$

Making the substitutions

$$m + \lambda x = \lambda x', \quad n + \lambda y = \lambda y',$$

and breaking up Ω into sets of homogeneous functions, its solution can be obtained at once. The corresponding class of ordinary differential equations is most easily solved in this way, and the transformation employed for its solution by Legendre (*Mémoires de l'Academie,* 1787) rendered unnecessary.

In the second class, the coefficients are symmetric functions of the *dependent* variable. Its type is

$$Az^{m-\alpha}\nabla^{\alpha}z + Bz^{m-(\alpha-1)}\nabla^{\alpha-1}z + \&c. = \Theta_n + \Theta_p + \&c.,$$

where Θ_n, Θ_p, &c. are homogeneous functions of x, y, of the degrees n, p, &c. respectively. Putting

$$z^m = z',$$

the equation is obviously reducible to the form

$$A'\nabla^{\alpha}z' + B'\nabla^{\alpha-1}z' + \&c. = \Theta_m + \Theta_n + \&c.;$$

the solution of which can be at once obtained by the method furnished in the paper before quoted.

Thus, the equation

$$z^{m-2}(x^2r + 2xys + y^2t) + Bz^{m-1}(xp + yq) + Cz^m = \Theta_n + \Theta_p$$

becomes

$$\left\{\frac{\nabla(\nabla - 1)}{m(m-1)} + B\frac{\nabla}{m} + C\right\} z^m = \Theta_n + \Theta_p.$$

Let the roots of

$$\nabla(\nabla - 1) + B(m-1)\nabla + Cm(m-1) = F(\nabla) = 0$$

be b and c, and the solution is

$$z^m = m(m-1)\left\{\frac{\Theta_n}{F(n)} + \frac{\Theta_p}{F(p)}\right\} + u_b + u_c,$$

where u_b, u_c are arbitrary homogeneous equations of the degrees b and c respectively.

We may derive from Gregory's *Examples* some pleasing illustrations of this method of solution. Thus, the equation

$$x\frac{dz}{dx} + y\frac{dz}{dy} = 2xy\sqrt{(a^2 - z^2)}$$

assumes the symbolic shape

$$\nabla \sin^{-1}\frac{z}{a} = 2xy;$$

and its solution is therefore

$$\frac{z}{a} = \sin(xy + u_0).$$

4. We may, in some cases, employ the Index Symbol to great advantage in the investigation of the solutions of systems of partial differential equations. The results exhibit themselves in a remarkably symmetrical and elegant form.

Thus, if we had the system

$$\left.\begin{aligned} r &= f_1(x, y), \\ s &= f_2(x, y), \\ t &= f_3(x, y), \end{aligned}\right\}$$

multiply the first equation by x^2, the second by $2xy$, the third by y^2, and adding, we get

$$x^2r + 2xys + y^2t = x^2f_1 + 2xyf_2 + y^2f_3.$$

Break up the right-hand member, as before, into sets of homogeneous functions, and the whole assumes the symbolic shape

$$\nabla(\nabla - 1)z = \Theta_m + \Theta_n + \Theta_p + \&c.;$$

and the required solution is

$$z = \frac{\Theta_m}{m(m-1)} + \frac{\Theta_n}{n(n-1)} + \&c. + u_0 + u_1,$$

where u_0, u_1 are arbitrary homogeneous functions of the degrees 0 and 1 respectively.

The apparent method of solving such a system would be, to integrate the first equation twice with respect to x, supposing y constant, thereby introducing two arbitrary functions of y; to integrate the second equation alternately with respect to x and y, thereby introducing two more arbitrary functions, the one of y and the other of x; and to integrate the third equation with respect to y twice, thereby introducing two more arbitrary functions of x. Finally, by a comparison of the solutions thus got, we should endeavour to determine the characters of the resultant arbitrary functions as far as possible.

It is obvious that our method of solution will apply to the system

$$\left.\begin{aligned} \frac{d^2z}{d\phi^2} &= f_1(e^\phi, e^\psi, \sin\phi, \sin\psi, \cos\phi, \cos\psi), \\ \frac{d^2z}{d\phi\, d\psi} &= f_2(e^\phi, e^\psi, \sin\phi, \sin\psi, \cos\phi, \cos\psi), \\ \frac{d^2z}{d\psi^2} &= f_3(e^\phi, e^\psi, \sin\phi, \sin\psi, \cos\phi, \cos\psi), \end{aligned}\right\},$$

as well as to many others which readily suggest themselves.

5. It is indispensable that we should discuss an exceptional case, which will sometimes occur in the employment of the present, as of any other, method of integration.

This arises from the circumstance that the inverse process may generate an infinite coefficient, and can be illustrated by the partial differential equation

$$x\frac{dz}{dx} + y\frac{dz}{dy} - az = \Theta_m.$$

The solution of this equation, as given by our method, is

$$z = \frac{\Theta_m}{m-a} + u_a;$$

in which, when $a = m$, the first term becomes *infinite*.

To clear away this difficulty, assume, in the general solution,

$$u_a = v_a - \frac{\Theta_a}{m-a},$$

which gives

$$z = \frac{\Theta_m - \Theta_a}{m-a} + v_a.$$

This becomes indeterminate when $a = m$; therefore, differentiating with respect to m both numerator and denominator, and remembering that

$$\Theta_m = \frac{x^m f\left(\frac{y}{x}\right) + y^m F\left(\frac{x}{y}\right)}{2},$$

we find for the solution *in the exceptional case*

$$z = \Theta_m\left(\frac{\log x + \log y}{2}\right) + v_m.$$

By an obvious extension it appears that the solution of

$$x\frac{dw}{dx} + y\frac{dw}{dy} + z\frac{dw}{dz} - mw = \Theta_m$$

is

$$w = \Theta_m\left(\frac{\log x + \log y + \log z}{3}\right) + v_m;$$

which, of course, may be generalized for n independent variables. Thus, the solution of

$$x\frac{dz}{dx} + y\frac{dz}{dy} = c$$

is

$$z = c\left(\frac{\log x + \log y}{2}\right) + u_0.$$

Again, the solution of

$$a\frac{dz}{dx}+b\frac{dz}{dy}=c$$

is
$$z=\frac{c}{2}\left(\frac{x}{a}+\frac{y}{b}\right)\phi_0(e^{\frac{x}{a}}, e^{\frac{y}{b}}).$$

The integral of this equation, as given by Gregory, is incomplete; and the same remark will apply to his solutions of the equations numbered in the *Examples* (7), (9), (26). Such a result might indeed be anticipated from the method of integration employed. It is, of course, *a priori* obvious that, as these partial differential equations are more or less symmetrical in x and y, their solutions should possess the same character.

Particular instances of the occurrence of such exceptional cases are furnished by the ordinary differential equations

$$\frac{dy}{d\theta}-ay=ce^{m\theta}, \quad \text{or} \quad x\frac{d}{dx}-ay=cx^m,$$

and $\dfrac{d^2y}{d\theta^2}+a^2y=\cos m\theta$, or $\left(x\dfrac{d}{dx}\right)^2 y+a^2y=\frac{1}{2}(x^{m\sqrt{(-1)}}+x^{-m\sqrt{(-1)}})$,

when $a=m$.

As regards this latter example, it may be observed that the partial differential equation corresponding is

$$\frac{d^2z}{d\theta^2}+2\frac{d^2z}{d\theta d\phi}+\frac{d^2z}{d\phi^2}+a^2z=\cos(m\theta+n\phi),$$

and its solution

$$z=\frac{\cos(m\theta+n\phi)}{a^2-(m+n)^2}+\psi_{a\sqrt{(-1)}}(e^\theta, e^\phi)+\psi_{-a\sqrt{(-1)}}(e^\theta, e^\phi).$$

The partial differential equation corresponding to the exceptional case is

$$\frac{d^2z}{d\theta^2}+2\frac{d^2z}{d\theta d\phi}+\frac{d^2z}{d\phi^2}+(m+n)^2z=\cos(m\theta+n\phi),$$

and its solution

$$z=\frac{\sin(m\theta+n\phi)}{2(m+n)}\cdot\frac{\theta+\phi}{2}+\psi_{(m+n)\sqrt{(-1)}}(e^\theta, e^\phi)+\psi_{-(m+n)\sqrt{(-1)}}(e^\theta, e^\phi).$$

The methods of integration hitherto in use seem inadequate to contend with the difficulties which the integration of such an equation would present.

Trinity College, Dublin,
October 1852.

SOLUTIONS OF TWO PROBLEMS.

By ARTHUR COHEN, Magdalene College, Cambridge.

THE following problem is not without historical interest, and the solution given in Gregory's *Examples* is rather complicated.

"To find a point within a given triangle, such that the sum of the distances of the point from the three vertices of the triangle may be a minimum."

Let (a_1b_1), (a_2b_2), (a_3b_3) be the coordinates of the three vertices A_1, A_2, A_3 of the triangle. Let x, y be the coordinates of the required point X. Let $XA_1 = r_1$, $XA_2 = r_2$, $XA_3 = r_3$.

Let θ_1, θ_2, θ_3 be the angles made by r_1, r_2, r_3, respectively, with the axis of x; and let, finally, ϕ_1, ϕ_2, ϕ_3 denote the angles between (r_2r_3), (r_3r_1), (r_1r_2), respectively, then

$$u=\sqrt{\{(x-a_1^2)+(y-b_1^2)\}}+\sqrt{\{(x-a_2^2)+(y-b_2^2)\}}+\sqrt{\{(x-a_3^2)+(y-b_3^2)\}}$$

is to be a minimum, therefore

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0,$$

therefore
$$\frac{x-a_1}{r_1} + \frac{x-a_2}{r_2} + \frac{x-a_3}{r_3} = 0,$$

$$\frac{y-b_1}{r_1} + \frac{y-b_2}{r_2} + \frac{y-b_3}{r_3} = 0,$$

therefore
$$\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0,$$

$$\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 0,$$

$$\therefore \sin(\theta_3 - \theta_1) + \sin(\theta_3 - \theta_2) = 0, \quad \therefore \phi_1 = \phi_2 = \phi_3;$$

and, more generally, if it be required to make $u = f(r_1, r_2, r_3)$ a maximum or minimum, we obtain, in a similar manner,

$$\frac{\dfrac{df}{dr_1}}{\sin\phi_1} = \frac{\dfrac{df}{dr_2}}{\sin\phi_2} = \frac{\dfrac{df}{dr_3}}{\sin\phi_3}.$$

The theorem that the geometrical mean of n positive quantities is less than their arithmetical mean has been proved in *Liouville's Journal* by means of the Differential Calculus. If, however, the Differential Calculus be employed, the following proof seems to be shorter than the one given in the French journal.

It may be easily proved that the *absolute* maximum value of $u = (x_1 \dots x_n)^{\frac{1}{n}}$ with the condition that $x_1 + \dots + x_n = c = a_1 + \dots + a_n$ is found by putting $x_1 = x_2 \dots = x_n$, and it is therefore $\frac{a_1 + \dots + a_n}{n}$, and any other value being less than the absolute maximum value, we have $(a_1 \dots a_n)^{\frac{1}{n}}$ is less than $\frac{a_1 + \dots + a_n}{n}$.

MATHEMATICAL NOTE.

By Professor De Morgan.

In a Letter of Newton to Collins, dated Nov. 8, 1676, there is so remarkable an assertion relative to the extent to which Newton had carried the integral calculus, that the little notice it has received is to be wondered at. Looking at the evidences which the *Principia* offers of the possession of more methods than were ever published, such an assertion is not to be lightly passed over, extraordinary as it may be.

The following extract was published by William Jones, in his *Analysis per Quantitatum Series, &c.*, Lond. 4to. 1711. The most interesting part of this work was reprinted in 1712 in the *Commercium Epistolicum*, so that it fell into comparative neglect. How what follows came to be omitted in the *Com. Epist.* it is difficult to say. Both in date and matter it would have been much to the purpose: it may be that Newton had subsequent doubts as to the correctness of the assertion in all its extent. The whole Letter is in the Macclesfield Collection (vol. II. pp. 403-5), to which the editor has added the following words: "An extract from

this Letter is published in the *Analysis per Quant. Series,* but with interpolations." This is not correct: the extract made by Jones (from which the following is taken) agrees to a syllable with what is printed in the Macclesfield Collection.

"There is no Curve-line express'd by any Equation of three terms, tho' the unknown quantities affect one another in it, or the Indices of their Dignities be surd quantities (suppose $ax^{\lambda} + bx^{\mu}y^{\sigma} + cy^{\tau} = 0$, where x signifies the Base, y the Ordinate, λ, μ, σ, τ the Indices of the Dignities of x and y, and a, b, c known quantities with their signs + or −,) I say, there is no such Curve-line, but I can, in less than half a quarter of an hour, tell whether it may be Squar'd, or what are the simplest Figures it may be compared with, be those Figures Conic Sections, or others. And then by a direct and short way (I dare say the shortest the nature of the thing admits of for a general one,) I can compare them. This may seem a bold assertion, because it's hard to say a Figure may, or may not, be Squar'd, or Compar'd with another; but it's plain to me by the fountain I draw it from, tho' I will not undertake to prove it to others. The same Method extends to Equations of four Terms, and others also, but not so generally."

Nov. 26, 1851.

PROBLEMS.

1. A VARIABLE ellipse always touches a fixed ellipse and has a common focus with it. Find the locus of the second focus—(1) when the axis-major is constant; (2) when the axis-minor is so.

2. Find the locus of the intersection of pairs of tangents to a cycloid which are at right angles to one another.

3. Prove the following expressions for the general differential coefficient of $\tan x$.

$$\frac{d^{2n}}{dx^{2n}} \tan x = \frac{(-1)^n \, 2^{2n+2} (1 + \Delta) \tan x}{4(1 + \Delta) + \Delta^2(1 + \tan^2 x)} \, 0^{2n},$$

$$\frac{d^{2n+1}}{dx^{2n+1}} \tan x = \frac{(-1)^n \, 2^{2n+1} \Delta (\Delta + 2) \, (1 + \tan^2 x)}{4 (1 + \Delta) + \Delta^2(1 + \tan^2 x)} \, 0^{2n+1}.$$

4. A perfectly rough sphere is laid upon a perfectly rough plane, inclined at a given angle to the horizon. This plane is then made to revolve uniformly about an axis perpendicular to itself. Determine the motion of the sphere.

5. (*a*) What would be the density and pressure of air at an immense distance from the earth, if the earth were at rest in a space of constant temperature?

(*b*) If the earth and moon were both at rest, at immense distances from one another, in a space of constant temperature, what would be the pressure and density at the surface of the moon? [Work out numerically, taking 0° centrigrade as the constant temperature; 2114 times the weight of a pound at the earth's surface, as the atmospheric pressure at the earth's surface; $\frac{1}{12{\cdot}383}$ lb. the mass of a cubic foot of air at 0° and under that pressure; 7912 miles the diameter of the earth; 2160 miles the diameter of the moon; ·163 the force of gravity at the moon's surface as compared with that at the surface of the earth.]

6. Find the attraction of a solid sphere of which the density varies inversely as the fifth power of the distance from an external point, on any point external or internal; the mutual attraction between two particles varying inversely as the square of their distance.

7. If a ball weighing W be shot from an air-gun, the volume of the barrel of which is U, by means of the expansion of a quantity of air which occupies the space V under the pressure P at the commencement of the motion, the mass of the air being very small compared with that of the ball, and the mass of the ball very small compared with that of the gun; shew that, provided the cooling effect of the expansion be not, during the motion of the ball through the barrel, sensibly compensated by the communication of any heat to the air from the matter round it, and provided there be no sensible loss of effect by friction, the velocity of the ball on leaving the gun is

$$\left\{\frac{2g}{W}\left[PV\frac{1}{k-1}\left\{1-\left(\frac{V}{U}\right)^{k-1}\right\}-\Pi U\right]\right\}^{\frac{1}{2}},$$

where Π denotes the atmospheric pressure, and k the ratio of the specific heat of air under constant pressure to the

specific heat of air in constant volume, which may be taken as a constant quantity. Find the proportion which the work spent in communicating the motion to the ball bears to that spent in producing noise and in overcoming fluid friction near the mouth of the gun.—*St. Peter's College Examination Papers.* Third Year. June 1852.

8. (*a*) If an infinite number of perfectly elastic material points equally distributed through a hollow sphere, be set in motion each with any velocity, shew that the resulting continuous pressure (referred to a unit of area) on the internal surface is equal to one-third of the *vis viva* of the particles divided by the volume of the sphere.—*St. Peter's College Examination Papers.* June 1852.

(*b*) Prove the same proposition for a hollow space of any form.

9. If in the case of homoloidal spaces, we denote the volume formed in space of r dimensions, from $r+1$ next inferior spaces by V; and through each of their $r+1$ points of intersection draw parallel lines meeting the opposite spaces in points, the volume formed from which is V_1; then

$$V_1 = (-1)^{r-1} r V.$$

If the lines pass through the same finite point, give the corresponding formula.

10. If (abc) denote the area of the triangle formed from the points abc, and (123_ρ) the area of the triangle formed by the polars with regard to a central conic of these points, divided by the radius of its circumscribing circle, and k^4 be the product of the squares of the semiaxes of the curve, and $O_1 O_2 O_3$ the perpendiculars from the centre on the polars; shew that

$$2 O_1 O_2 O_3 (abc) = k^4 (123_\rho).$$

Hence, by means of the expression for the radius of curvature of a conic, compare in the limit the radii of the circles circumscribing the 3 triangles formed from 3 points on the curve, their tangents, and one chord and the tangents at its extremities; also the areas of the 2 first triangles.

Evaluate the expression when one of the points is the centre of the conic. What is the corresponding form for general space?

ON THE RATIONALISATION OF CERTAIN ALGEBRAICAL EQUATIONS.

By Arthur Cayley.

Suppose $$x + y = 0, \quad x^2 = a, \quad y^2 = b;$$
then if we multiply the first equation by 1, xy, and reduce by the two others, we have

$$x + y = 0,$$
$$bx + ay = 0,$$

from which, eliminating x, y,

$$\begin{vmatrix} 1, & 1 \\ b, & a \end{vmatrix} = 0;$$

which is the equation between a and b, Or, considering x, y as quadratic radicals, the rational equation between x, y. So if the original equation be multiplied by x, y, we have

$$a + xy = 0,$$
$$b + xy = 0.$$

Or, eliminating 1, xy,

$$\begin{vmatrix} a, & 1 \\ b, & 1 \end{vmatrix} = 0,$$

which may be in like manner considered as the rational equation between x, y.

The preceding results are of course self-evident, but by applying the same process to the equations

$$x + y + z = 0, \quad x^2 = a, \quad y^2 = b, \quad z^2 + c,$$

we have results of some elegance. Multiply the equation first by 1, yz, zx, xy, reduce and eliminate the quantities x, y, z, xyz, we have the rational equation

$$\begin{vmatrix} & 1 & 1 & 1 \\ 1 & . & c & b \\ 1 & c & . & a \\ 1 & b & a & . \end{vmatrix} = 0.$$

Again, multiply the equation by x, y, z, xyz, reduce and eliminate the quantities 1, yz, zx, xy, the result is

$$\begin{vmatrix} & a & b & c \\ a & . & 1 & 1 \\ b & 1 & . & 1 \\ c & 1 & 1 & . \end{vmatrix} = 0.$$

which is of course equivalent to the preceding one (the two determinants are in fact identical in value), but the form is essentially different. The former of the two forms is that given in my paper "On a theorem in the Geometry of Position," (Old Series, vol. II. p. 270): it was only very recently that I perceived that a similar process led to the latter of the two forms.

Similarly, if we have the equations

$$x + y + z + w = 0, \quad x^2 = a, \quad y^2 = b, \quad z^2 = c, \quad w^2 = d,$$

then multiplying by 1, yz, zx, xy, xw, yw, zw, $xyzw$, reducing and eliminating the quantities,

we have the result

$$\begin{array}{c} \begin{array}{cccccccc} x, & y, & z, & w, & yzw, & zwx, & wxy, & xyz \end{array} \\ \left|\begin{array}{ccc|c|ccc|c} 1 & 1 & 1 & 1 & . & . & . & . \\ \hline . & c & b & . & 1 & . & . & 1 \\ c & . & a & . & . & 1 & . & 1 \\ b & a & . & . & . & . & 1 & 1 \\ \hline d & . & . & a & . & 1 & 1 & \\ . & d & . & b & 1 & . & 1 & \\ . & . & d & c & 1 & 1 & . & \\ \hline . & . & . & . & a & b & c & d \end{array}\right| \end{array} = 0.$$

So if we multiply the equations by x, y, z, w, yzw, zwx, wxy, and xyz, reduce and eliminate the quantities,

we have the result

$$\begin{array}{c} \begin{array}{cccccccc} 1, & yz, & zx, & xy, & xw, & yw, & zw, & xyzw \end{array} \\ \left|\begin{array}{c|ccc|ccc|c} a & . & 1 & 1 & 1 & . & . & . \\ b & 1 & . & 1 & . & 1 & . & . \\ c & 1 & 1 & . & . & . & 1 & . \\ \hline d & . & . & . & 1 & 1 & 1 & . \\ \hline . & d & . & . & . & c & b & 1 \\ . & . & d & . & c & . & a & 1 \\ . & . & . & d & b & a & . & 1 \\ \hline . & a & b & c & . & . & . & 1 \end{array}\right| \end{array} = 0.$$

which however is not essentially distinct from the form before obtained, but may be derived from it by an interchange of lines and columns.

And in general for any *even* number of quadratic radicals the two forms are not essentially distinct, but may be derived from each other by interchanging lines and columns, while for an *odd* number of quadratic radicals the two forms cannot be so derived from each other, but are essentially distinct.

I was indebted to Mr. Sylvester for the remark that the above process applies to radicals of a higher order than the second. To take the simplest case, suppose

$$x + y = 0, \quad x^3 = a, \quad y^3 = b.$$

And multiply first by $1, x^2y, xy^2$; this gives

$$\begin{array}{llll} x & + \; y & & = 0 \\ . & \;\; ay & + \; x^2y^2 & = 0 \\ bx & \;\; . & + \; x^2y^2 & = 0\,; \end{array}$$

or, eliminating,

$$\begin{vmatrix} 1 & 1 & . \\ . & a & 1 \\ b & . & 1 \end{vmatrix} = 0.$$

Next multiply by x, y, x^2y^2; this gives

$$\begin{array}{llll} x^2 & \;\; . & + \; xy & = 0 \\ . & \;\; y^2 & + \; xy & = 0 \\ bx^3 & + \; ay^2 & \;\; . & = 0\,; \end{array}$$

or, eliminating,

$$\begin{vmatrix} 1 & . & 1 \\ . & 1 & 1 \\ b & a & . \end{vmatrix} = 0.$$

And lastly, multiply by x^2, y^2, xy; this gives

$$\begin{array}{llll} a & + \; x^2y & \;\; . & = 0 \\ b & \;\; . & + \; xy^2 & = 0 \\ . & \;\; x^2y & + \; xy^2 & = 0\,; \end{array}$$

or, eliminating,

$$\begin{vmatrix} a & 1 & . \\ b & . & 1 \\ . & 1 & 1 \end{vmatrix} = 0.$$

And it is proper to remark that the second and third forms are not essentially distinct, since the one may be derived from the other by the interchange of lines and columns.

Apply the preceding process to the system

$$x + y + z = 0, \quad x^3 = a, \quad y^3 = b, \quad z^3 = c.$$

First multiply by 1, xyz, $x^2y^2z^2$, x^2z, y^2x, z^2y, x^2y, y^2z, z^2x, reduce and eliminate the quantities,

$$\text{the result is}\quad \begin{array}{ccc|ccc|ccc} x, & y, & z, & y^2z^2, & x^2yz, & y^2zx, & z^2xy, & z^2x^2, & x^2y^2 \\ \hline 1 & 1 & 1 & & & & & & \\ \hline & & & 1 & 1 & 1 & & & \\ \hline & & & & & & a & b & c \\ \hline \cdot & \cdot & a & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ b & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & c & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \hline \cdot & a & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & b & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ c & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \end{array} = 0.$$

Next multiply by x, y, z, y^2z^2, z^2x^2, x^2y^2, x^2yz, y^2zx, z^2xy, reduce and eliminate the quantities,

$$\text{the result is}\quad \begin{array}{ccc|ccc|ccc} x^2, & y^2, & z^2, & yz, & zx, & xy, & xy^2z^2, & yz^2x^2, & zx^2y^2 \\ \hline 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & c & b & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ c & \cdot & a & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ b & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \hline \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & b & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & 1 & 1 & \cdot \end{array} = 0.$$

Lastly, multiply by x^2, y^2, z^2, yz, zx, xy, xy^2z^2, yz^2x^2, xy^2z^2, reduce and eliminate the quantities,

$$\text{the result is}\quad \begin{array}{c|c|c|ccc|ccc} 1 & xyz, & x^2y^2z^2, & yz^2, & zx^2, & xy^2, & y^2z, & z^2x, & x^2y \\ \hline a & & & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ b & & & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ c & & & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \hline & 1 & & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ & 1 & & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ & 1 & & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \hline & & 1 & \cdot & \cdot & c & \cdot & b & \cdot \\ & & 1 & a & \cdot & \cdot & \cdot & \cdot & c \\ & & 1 & \cdot & b & \cdot & a & \cdot & \cdot \end{array} = 0,$$

where, as in the case of two cubic radicals, two forms, viz. the first and third forms of the rational equation, are not essentially distinct, but may be derived from each other by interchanging lines and columns.

And in general, whatever be the number of cubic radicals, two of the three forms are not essentially distinct, but may be derived from each other by interchanging lines and columns.

2, *Stone Buildings,*
Dec. 28, 1852.

NOTE ON THE DOCTRINE OF IMPOSSIBLES.

By WILLIAM WALTON.

IN a note in page 47 of the last Number of the *Journal,* Mr. Salmon has made certain observations on the question of Geometrical Impossibles, in reply to my paper on the subject in the previous Number. I subjoin in separate paragraphs, for convenience of reference, all his remarks except one, which does not seem to me to bear upon the question at issue.

(1). "I see no reason why we should not close our controversy on the terms of arbitration proposed by Professor De Morgan, namely that Mr. Gregory's conventions shall be banished from the regions of Algebraic Geometry to those of Geometrical Algebra."

(2). "Mr. Walton does not deny the only point for which I am anxious to contend, viz. that the curvilinear loci obtained by Mr. Gregory's rules have *no geometrical connection* with plane curves represented by the same equations."

(3). "And if this be so, they cannot be expected to throw any light on any difficulty, real or supposed, in the theory of plane curves."

(4). "I have only to add that I believe Mr. Walton was hasty in asserting (vol. VII. p. 239) that if $f(x, y) = 0$ be transcendental, a conjugate point, not double but single, may easily present itself."

(5). "And that the case of a conjugate point appearing to have a real tangent is explained by observing that such a point results from the union of two or more ordinary conjugate points."

(1). I quite concur with Mr. Salmon in thinking that we may close our controversy on the terms of arbitration proposed by Professor De Morgan, whose excellent term *Geometrical Algebra* so exactly accords with the distinction, on which I myself most carefully insisted in my own paper, between the two provinces of reasoning.

(2). I neither assert nor deny that there is any *geometrical connection* between the possible and the impossible branches of the locus of an equation involving x and y; conceiving that the one or the other view might be adopted according to the precise sense attached to the phrase.

(3). I am quite ready to admit that all the properties of the possible branches may be thoroughly discussed without any reference to the impossible ones, even where I might not myself see any obvious explanation of an apparent difficulty; just as I should believe it possible to examine adequately the nature of the impossible ones without considering the possible.

(4). Take the equation

$$x + y = x^2\{\alpha^{x\sqrt{(-1)}} + \alpha_1^{x\sqrt{(-1)}} + \alpha_2^{x\sqrt{(-1)}} + \dots + \alpha_n^{x\sqrt{(-1)}}\}.$$

There is a *single* conjugate point at the origin, the equation to the tangent of the one branch through it being $x + y = 0$.

The only objections which, as far as I am aware, could be urged against what I have here said, are either that an equation to a curve cannot be admitted to involve the symbol $\sqrt{(-1)}$, or that $\sqrt{(-1)}$ is necessarily equivalent to $\pm\sqrt{(-1)}$. The former objection would be at variance with the object of *Geometrical Algebra,* which professes to trace out a curvilinear locus for any equation whatever in x and y: those who might entertain the latter objection I must refer to a paper by Gregory on the Impossible Logarithms of Quantities, in the *Cambridge Mathematical Journal,* vol. I. p. 226.

I do not say that a *single* conjugate point may not exist in curves not transcendental: take for instance the equation

$$x + y = x^2\sqrt{(-1)}.$$

I specified transcendental equations in particular owing to its being necessary, provided that we desire by transposition of terms, squaring, &c. to get rid of the symbol $\sqrt{(-1)}$, to have recourse to infinite series or their symbols.

(5). I thank Mr. Salmon for his suggestion that the existence of a real tangent at a conjugate point may be

explained by the idea of the union of two or more conjugate points, as it is always I think interesting, in philosophical inquiries, to contemplate the same phenomenon from different points of view.

Cambridge, Jan. 15, 1853.

ON CERTAIN GEOMETRICAL THEOREMS.

By W. Spottiswoode.

Theorem I. *When two systems of four points taken upon two straight lines, and corresponding each to each, have their anharmonic ratios equal, if the two straight lines be so placed that two homologous points coincide, the three straight lines, which join the three other points of the first system to the three homologous points of the second respectively, will meet in a point.*

Let $0, x_1, x_2, x_3$ be the distances of the four points on the first line, and $0, y_1, y_2, y_3$ those on the second, measured from the point of intersection; then the equivalence of the anharmonic ratios of the two systems will be thus expressed:

$$\begin{vmatrix} \frac{1}{x_1} & \frac{1}{y_1} & 1 \\ \frac{1}{x_2} & \frac{1}{y_2} & 1 \\ \frac{1}{x_3} & \frac{1}{y_3} & 1 \end{vmatrix} = 0.$$

But if the two straight lines be taken as coordinate axes, this equation expresses the condition that the three straight lines

$$\frac{x}{x_1} + \frac{y}{y_1} - 1 = 0,$$

$$\frac{x}{x_2} + \frac{y}{y_2} - 1 = 0,$$

$$\frac{x}{x_3} + \frac{y}{y_3} - 1 = 0,$$

shall meet in a point.

But these lines pass through the pairs of points

$$0,\ x_1;\ 0,\ y_1,$$
$$0,\ x_2;\ 0,\ y_2,$$
$$0,\ x_3;\ 0,\ y_3,$$

respectively. Hence the theorem above enunciated.

THEOREM II. *When two systems of four straight lines, which correspond each to each respectively, have their anharmonic ratios equal, if they be so placed that two corresponding straight lines coincide in direction, the three other straight lines of the first pencil will meet the corresponding lines of the second respectively in three points situated on the same straight line.*

If ω, α, β, γ be the four straight lines forming the first pencil, and ω, α_1, β_1, γ_1 those forming the second, the equivalence of the anharmonic ratios of the two pencils will be thus expressed:

$$\begin{vmatrix} \cot\omega\alpha, & \cot\omega\alpha_1, & 1 \\ \cot\omega\beta, & \cot\omega\beta_1, & 1 \\ \cot\omega\gamma, & \cot\omega\gamma_1, & 1 \end{vmatrix} = 0.$$

But if the common straight line ω be taken as the axis of x, and the centre of one pencil as the origin; then, a being the distance between the two centres, and x_1, y_1; x_2, y_2; x_3, y_3 being the coordinates of the points of intersection of α, α_1; β, β_1; γ, γ_1, respectively, on substituting for $\cot\omega\alpha, \ldots$, $\cot\omega\alpha_1, \ldots$ in terms of $x_1, y_1, \ldots$ and dividing out the term a common to an entire vertical row, the above determinant becomes

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$

which is the condition that the three points of intersection shall lie on a straight line. Q. E. D.

ON CERTAIN GEOMETRICAL RELATIONS BETWEEN A SURFACE OF THE SECOND DEGREE AND A TETRAHEDRON WHOSE EDGES TOUCH THE SURFACE.

By Thomas Weddle.

Let $t = 0$, $u = 0$, $v = 0$, $w = 0$, be the equations to the faces of a tetrahedron; then if t, u, v, w have been multiplied by arbitrary constants, the equation to any surface of the second degree may be denoted by

$$t^2 + u^2 + v^2 + w^2 + 2\lambda tu + 2\mu tv + 2\nu tw + 2\rho uv + 2\sigma uw + 2\tau vw = 0.$$

If the surface which this equation denotes touch the edge (vw) of the tetrahedron, then, putting $v = w = 0$ in the preceding equation, the result $t^2 + u^2 + 2\lambda tu = 0$, must be a complete square, and this requires $\lambda = \pm 1$. Similarly, that the surface may touch the other edges, we must have $\mu = \pm 1$, $\nu = \pm 1$, $\rho = \pm 1$, $\sigma = \pm 1$, and $\tau = \pm 1$; and the equation to the surface becomes

$$t^2 + u^2 + v^2 + w^2 \pm 2tu \pm 2tv \pm 2tw \pm 2uv \pm 2uw \pm 2vw = 0 \ldots (1).$$

Now the double signs in this equation, being independent of each other, may be combined in $(2^6 =)$ 64 different ways; but it will on examination be found that there are eight ways in which the signs of (1) may be taken so that the left-hand member shall be a complete square, and the equation will then denote a plane; also there are 48 ways in which the signs of (1) may be combined so that it shall denote a cone having its vertex on one of the edges of the tetrahedron, (thus $t^2 + u^2 + v^2 + w^2 + 2tu - 2tv - 2tw - 2uv - 2uw - 2vw = 0$, or $(-t - u + v + w)^2 + (v - w)^2 = (v + w)^2$, denotes a cone having its vertex at the point $t + u = v = w = 0$, on the edge (vw)); hence these $(8 + 48 =)$ 56 combinations of signs must be rejected, and there remain $(64 - 56 =)$ 8 combinations to be considered. One of these gives

$$t^2 + u^2 + v^2 + w^2 - 2tu - 2tv - 2tw - 2uv - 2uw - 2vw = 0 \ldots (2);$$

and if this be denoted by $f(t, u, v, w) = 0$, it will be found that the other seven combinations are

$$f(-t, -u, v, w) = 0,\ f(-t, u, -v, w) = 0,\ f(-t, u, v, -w) = 0,$$
$$f(-t, u, v, w) = 0,\quad f(t, -u, v, w) = 0,\quad f(t, u, -v, w) = 0,$$
$$\text{and } f(t, u, v, -w) = 0:$$

but since each of these may, by changing the signs of one or more of the quantities t, u, v, w, be reduced to $f(t, u, v, w) = 0$,

we conclude that (2) is the most general form of the equation to surfaces of the second degree touching the edges of the tetrahedron. It must be remembered that since t, u, v and w have been supposed multiplied by arbitrary constants, (2) implicitly contains three arbitrary constants, as it evidently ought.*

Let A, B, C and D be the angular points of the tetrahedron, the faces BCD, CDA, DAB, and ABC being denoted by $t=0$, $u=0$, $v=0$, and $w=0$, respectively; also let E, F, G, e, f, g be the points of contact of the surface (2) with the edges AB, AC, AD, CD, BD, and BC, respectively; and e_1, f_1, g_1, E_1, F_1, G_1, the points in which tangent planes to (2) that pass through these edges intersect the opposite edges; so that E and E_1 are on the edge AB, &c.

To find the equations to the point E, put $v=w=0$ in (2), and we have $t^2+u^2-2tu=0$, or $t-u=0$. In a similar manner we shall find that the points E, F, G, e, f, g are denoted as follows:

$$E, \quad v=w=t-u=0 \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$F, \quad w=u=t-v=0 \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$G, \quad u=v=t-w=0 \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

---

* If the surface, instead of touching all the edges of the tetrahedron, only touch the four edges (vw), (wt), (tu), (uv); that is, the sides AB, BC, CD, DA, of a twisted quadrilateral $ABCD$, its equation will be

$$t^2+u^2+v^2+w^2\pm 2tu\pm 2uv\pm 2vw\pm 2tw+2\mu tv+2\sigma uw=0;$$

or, since we may change the sign of any of the quantities u, v, w, the equation may be written

$$t^2+u^2+v^2+w^2-2tu-2uv-2vw\pm 2tw+2\mu tv+2\sigma uw=0 \ldots\ldots (\alpha).$$

Hence the points H, K, L, M of contact of the sides AB, BC, CD, DA, respectively, will be denoted as follows:

$$\left.\begin{array}{ll} H, & v=w=t-u=0 \\ K, & w=t=u-v=0 \\ L, & t=u=v-w=0 \\ M, & u=v=w\pm t=0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(\beta).$$

Now the points HKL are situated in the plane $t-u+v-w=0$; and if the lower sign in (β) be taken, this plane also contains the point M. If, however, the upper sign be taken, let the plane HKL intersect the side (uv) or DA in the point N: now since the harmonic planes $w=0$, $t=0$, $w+t=0$, and $w-t=0$, pass through the points A, D, M, and N, respectively, it follows that the side DA is divided harmonically in M and N. Hence

If a surface of the second degree touch the sides of a twisted quadrilateral, the four points of contact will either be in one plane; or else, any side of the quadrilateral will be divided harmonically by its point of contact and its intersection with the plane of the other three points of contact.

e, $t = u = v - w = 0$ ....(6),

f, $t = v = w - u = 0$ ....(7),

g, $t = w = u - v = 0$ ....(8).

Hence the equations to the planes drawn through the edges of the tetrahedron and the points of contact of the opposite edges are as under.

Equation to the plane CDE, $t - u = 0$ .... (9),

BDF, $t - v = 0$..................(10),

BCG, $t - w = 0$..................(11),

ABe, $v - w = 0$..................(12),

ACf, $w - u = 0$..................(13),

ADg, $u - v = 0$..................(14).

Again, since the equation (2) may be put under the form

$$(t-u)^2 + (v-w)^2 = 2(t+u)(v+w) \text{ ....}(\gamma),$$

it appears that $t + u = 0$, and $v + w = 0$, denote tangent planes, and they evidently pass through the edges CD and AB respectively. In a similar manner the tangent planes through the other edges may be obtained as under.

The equation to the tangent plane through the edge

AB touching at E is $v + w = 0$.........(15),

AC .... F ... $w + u = 0$.........(16),

AD .... G ... $u + v = 0$.........(17),

CD .... e ... $t + u = 0$.........(18),

BD .... f ... $t + v = 0$.........(19),

BC .... g ... $t + w = 0$.........(20).

Hence the points E_1, F_1, &c. are denoted as follows:

E_1, $v = w = t + u = 0$...............(21),

F_1, $w = u = t + v = 0$...............(22),

G_1, $u = v = t + w = 0$...............(23),

e_1, $t = u = v + w = 0$...............(24),

f_1, $t = v = w + u = 0$...............(25),

g_1, $t = w = u + v = 0$...............(26).

It is evident from equation (γ) that the tangent plane $v + w = 0$ meets the surface (γ); that is, (2), in *one* point only; hence the surface cannot be either an hyperboloid

of one sheet, an hyperbolic paraboloid, a cone, or a cylinder, but must be either an ellipsoid, an hyperboloid of two sheets, or an elliptic paraboloid. In other words,

I. *No ruled or developable* surface of the second degree can touch* ALL *the edges of a tetrahedron, so that every surface of the second degree touching all the edges is necessarily umbilical.*

In my first memoir "On the Theorems in Space analogous to those of Pascal and Brianchon in a Plane," (*Journal*, new series, vol. IV. pp. 30 and 33), I have also shewn that no ruled surface of the second degree can touch all the edges of either a hexahedron or an octahedron. I have not as yet met with an instance in which a ruled surface of the second degree touches all the edges of a solid figure.

It appears from what has been already said, that (1) denotes 64 surfaces; 8 of which are planes, 8 umbilical surfaces, and 48 cones; and it is easily seen that the whole of these surfaces meet the edges of the tetrahedron in the points E, E_1; e, e_1; F, F_1; f, f_1; G, G_1; g, g_1 only—each surface meeting the edges in some six of these points. Now, taking one point on each edge, the preceding six pairs of points can be combined, six in each combination, in ($2^6 =$) 64 different ways (and no more), and therefore these different ways correspond to the different surfaces just mentioned. Hence

II. *If any six of the points* E, E_1; e, e_1; F, F_1; f, f_1; G, G_1; g, g_1—*one point on each edge—be taken, then*

either *the six points are in one plane;*

or, *are the points of contact of an umbilical surface of the second degree touching the edges of the tetrahedron;*

or, *one of the points is the vertex of a cone of the second degree which touches five of the edges in the remaining points.*

There are, as already observed, eight ways in which a selection of six points may be made so that they shall be in one plane; the equations of these eight planes and the points through which they pass are easily seen to be as under:

$$EFG\mathrm{e}_1\mathrm{f}_1\mathrm{g}_1, \quad -t+u+v+w=0 \text{............(27)},$$

$$E\mathrm{fge}_1F_1G_1, \quad t-u+v+w=0 \text{............(28)},$$

$$\mathrm{e}F\mathrm{g}E_1\mathrm{f}_1G_1, \quad t+u-v+w=0 \text{............(29)},$$

$$\mathrm{ef}GE_1F_1\mathrm{g}_1, \quad t+u+v-w=0 \text{............(30)},$$

* We have seen however that it is possible for a cone to touch five of the edges, and to have its vertex on the sixth edge. Such a cone may, if we please, be considered to touch all the edges.

$E_1F_1G_1e_1f_1g_1$, $t + u + v + w = 0$............(31),

E_1FGe_1fg, $-t - u + v + w = 0$............(32),

EF_1Gef_1g, $-t + u - v + w = 0$............(33),

EFG_1efg_1, $-t + u + v - w = 0$............(34).

It will be observed that the four planes (27), (28), (29), and (30), pass through the points of contact on the contiguous edges of the tetrahedron.

Put the left-hand members of (27),...(34) equal to $2t'$, $2u'$, $2v'$, $2w'$, $2p$, $2q$, $2r$, and $2s$, respectively, and it will readily be found that we have the following identities:

$$\left.\begin{array}{lllll}
2p & = & t + u + v + w & = t' + u' + v' + w' \\
2q & = & -t - u + v + w & = t' + u' - v' - w' \\
2r & = & -t + u - v + w & = t' - u' + v' - w' \\
2s & = & -t + u + v - w & = t' - u' - v' + w' \\
2t' & = & -t + u + v + w & = p + q + r + s \\
2u' & = & t - u + v + w & = p + q - r - s \\
2v' & = & t + u - v + w & = p - q + r - s \\
2w' & = & t + u + v - w & = p - q - r + s \\
2t & = & -t' + u' + v' + w' & = p - q - r - s \\
2u & = & t' - u' + v' + w' & = p - q + r + s \\
2v & = & t' + u' - v' + w' & = p + q - r + s \\
2w & = & t' + u' + v' - w' & = p + q + r - s
\end{array}\right\}\ldots(35).$$

The equation (2) to the surface touching the edges of the tetrahedron may also be written in any of the three following forms:

$$tt' + uu' + vv' + ww' = 0 \ldots\ldots\ldots\ldots\ldots(36),$$

$$t'^2 + u'^2 + v'^2 + w'^2 - 2t'u' - 2t'v' - 2t'w' - 2u'v' - 2u'w' - 2v'w' = 0 \ldots(37),$$

$$q^2 + r^2 + s^2 = p^2 \ldots\ldots\ldots\ldots\ldots\ldots(38).\text{*}$$

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* If $ap + bq + cr + es = 0$..........................(α)

denote a tangent plane to (38), then will

$$b^2 + c^2 + e^2 = a^2 \ldots\ldots\ldots\ldots\ldots(\beta),$$

and the point of contact will be determined by the equations

$$-\frac{p}{a} = \frac{q}{b} = \frac{r}{c} = \frac{s}{e} \ldots\ldots\ldots\ldots\ldots(\gamma).$$

And conversely, if (γ) denote a point in the surface (38), so that we have the condition (β), then will (α) denote the tangent plane at that point.

The planes t', u', v', w' will be the faces of a second tetrahedron $A'B'C'D'$; and the planes p, q, r, s those of a third tetrahedron $OO'O''O'''$; the faces $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$, $O'O''O'''$, $O''O'''O$, $O'''OO'$, and $OO'O''$, being those denoted by $t' = 0$, $u' = 0$, $v' = 0$, $w' = 0$, $p = 0$, $q = 0$, $r = 0$, and $s = 0$, respectively. I shall refer to the tetrahedra $ABCD$, $A'B'C'D'$, and $OO'O''O'''$, as the given, second, and third tetrahedra respectively; and when two tetrahedra are referred to without specifying which, I mean the first two. Omitting the third tetrahedron for the present, I proceed to the consideration of properties connected with the other two.

At the point A' we have $u' = v' = w' = 0$, or $-t = u = v = w$; and at O''' we have $p = q = r = 0$, or $-t = u = v = -w$, and so on; hence the angular points of the second and third tetrahedra are denoted as follows:

$$\left.\begin{array}{ll} A', & -t = u = v = w \\ B', & t = -u = v = w \\ C', & t = u = -v = w \\ D', & t = u = v = -w \end{array}\right\} \ldots\ldots\ldots\ldots(39),$$

$$\left.\begin{array}{ll} O, & t = u = v = w \\ O', & -t = -u = v = w \\ O'', & -t = u = -v = w \\ O''', & -t = u = v = -w \end{array}\right\} \ldots\ldots\ldots\ldots(40).$$

The intersection of each pair of tangent planes (15, 18), (16, 19), and (17, 20) is obviously situated in the plane $t + u + v + w = 0$; hence

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Similarly, if we take the more general equation

$$kp^2 + lq^2 + mr^2 + ns^2 = 0,$$

it may be shewn that if

$$ap + bq + cr + es = 0$$

be the equation to a tangent plane, then

$$k^{-1}a^2 + l^{-1}b^2 + m^{-1}c^2 + n^{-1}e^2 = 0,$$

and the point of contact will be denoted by

$$\frac{kp}{a} = \frac{lq}{b} = \frac{mr}{c} = \frac{ns}{e};$$

and conversely.

These formulas are capable of easy extension, but it is unnecessary to say more here, as I shall treat the subject in a much more general manner before I conclude.

III. *If a surface of the second degree touch the edges of a tetrahedron, the tangent planes passing through the opposite edges will intersect in three straight lines in one plane.*

Moreover, since the six planes (9...14) all pass through the point $t = u = v = w$, we infer that

IV. *The six planes which pass through the edges of the given tetrahedron and the points of contact of the opposite edges, intersect in one point;* or, which is the same thing, *the three straight lines joining the points of contact of opposite edges pass through one point.*

Since the planes $u = 0$, $t = 0$, $t - u = 0$, $t + u = 0$, form a harmonic system, and pass through the points A, B, E, E_1, on the edge AB, it appears that

V. *The two faces and the tangent plane through any edge, together with the plane passing through the same edge and the point of contact of the opposite edge, form a harmonic system; and hence any edge is divided harmonically by its point of contact and the point in which it is intersected by the tangent plane through the opposite edge.**

Again, (27...30) and (35), the straight lines (tt'), (uu'), (vv') and (ww') are all situated in the plane $t + u + v + w = 0$; hence

VI. *If (four) planes be drawn through the points of contact of every three contiguous edges of the given tetrahedron, these planes will intersect the corresponding faces of the tetrahedron in four straight lines in one plane.* This amounts to saying that, *the corresponding faces of the two tetrahedra $ABCD$, $A'B'C'D'$ intersect in four straight lines in one plane.*

The equations to the straight lines AA', BB', CC', and DD' are, (39),

$$\begin{aligned} u &= v = w \\ v &= w = t \\ w &= t = u \\ \text{and}\quad t &= u = v, \end{aligned}$$

and these pass through the point $t = u = v = w$ or O. Consequently

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* Hence when the points of contact are given, the tangent planes passing through the edges may be constructed.

It is evident, however, that the six points of contact are not all independent, but that if three of them (which must not be all in the same face of the tetrahedron) be given, the other three can be found. This may be done in various ways, one of which is derived from the well-known plane theorem, that "If a triangle be circumscribed about a conic, the three straight lines joining the points of contact to the opposite angles intersect in a point," so that when two of the points of contact are given, the third may be readily found. We have only to apply the construction here indicated to three of the faces of the tetrahedron, to find the points of contact of the other three edges.

VII. *The straight lines joining the corresponding angles of the tetrahedra ABCD, A′B′C′D′ intersect in one point.**

On account of the complete reciprocity of t, u, v, w and t', u', v', w' (see 2, 37, 35), the following is evident:

VIII. *The edges of the tetrahedron A′B′C′D′ touch the surface* (2), *and the points of contact coincide with those of the edges of the tetrahedron ABCD.*

Since the equations to the edge $C'D'$ are $t' = u' = 0$, or (35) $t - u = v + w = 0$, it follows that the edge $C'D'$ intersects the edge CD in $t = u = v + w = 0$, or the point e_1; and likewise the edge AB in $v = w = t - u = 0$, or the point E. Hence

IX. *The edges A′B′, A′C′, A′D′, C′D′, D′B′, B′C′ intersect the edges AB, AC, AD, CD, DB, BC, respectively, in the points* E_1, F_1, G_1, e_1, f_1, g_1; *and the former edges likewise intersect the edges CD, DB, BC, AB, AC, AD, respectively, in the points of contact* e, f, g, *E, F, G. The former six pairs of edges lie in the planes* (9...14), *and the latter six pairs in the tangent planes* (15...20).

The equations to the lines Be, Cf, Dg are, (6, 7, 8), $t = 0$, combined with $v - w = 0$, $w - u = 0$, and $u - v = 0$, respectively; and hence these lines pass through the point in which the face BCD (whose equation is $t = 0$) is intersected by the line AA' (whose equations are $u = v = w$). In a similar manner it may be shewn that the lines $B'E$, $C'F$, and $D'G$ intersect in the point in which the same line AA' intersects the face $B'C'D'$. Hence

X. *In any face of either of the tetrahedra ABCD, A′B′C′D′, let lines be drawn from the angles of the triangle which constitutes that face to the points of contact of the opposite sides, then shall these three lines pass through the point in which the said face is intersected by the line joining the corresponding angular points of the tetrahedra.*

In Plane Geometry we have the following theorem: "If a triangle be circumscribed about a conic, the straight lines joining the angles and the points of contact of the opposite sides intersect in a point"; and it is easy to recognise that (IV.) is analogous to this theorem; as is also theorem (XII.) (due to M. Chasles) in my third memoir "On the Theorems in Space analogous to Pascal and Brianchon in a Plane,"

* It appears from (18, 19, 20) that the three tangent planes through the edges BC, CD and DB in the face BCD intersect in the point $-t = u = v = w$ or A'; and similarly it may be shewn that the tangent planes through the edges in the other faces intersect in the points B', C', and D'. It is hence easily seen that theorems (VI.) and (VII.) justify the foot-note at p. 131, vol. VI. of this *Journal*.

(*Journal*, new series, vol. VI. p. 123). So likewise is the following: "If a surface of the second order be tangential to three planes, the planes passing through the mutual intersections of every two of them, and the point of contact of the third tangent plane, will intersect in the same straight line."* The preceding are not the only analogous properties however. (VII.) may be viewed as an analogue, if we present the plane theorem in this form: "If a triangle be circumscribed about a conic and another be inscribed in the same, having its angles at the points of contact of the sides of the former, then shall the straight lines joining the corresponding angular points intersect in one point." I dare say the analogy between (VII.) and this theorem will not be denied by most mathematicians, but in case there should be any doubt, I give one way of establishing an analogy. If a plane figure be circumscribed about a conic, and the points of contact of the contiguous sides be joined, a plane figure will be inscribed in the conic. Now if the edges of a solid figure touch a surface of the second degree, and planes be drawn through the points of contact of the contiguous edges, these planes will form the faces of another solid figure whose edges will also touch the surface; and the two solids will be such that the faces of each pass through the points of contact of the contiguous edges of the other. It hence follows that two solids so related bear some analogy to two plane figures, one inscribed in, and the other circumscribed about a conic, the sides of the latter figure touching at the angular points of the former.

I wish to state here that theorem (IV.), and the general equation (2) to surfaces of the second degree touching the edges of a tetrahedron, were discovered by the late G. W. Hearn and myself, independently of each other, and about the same time. It also appears (from letters in my possession) that Mr. Hearn had also investigated many of the preceding equations and theorems (and possibly some of those that follow), but he never pointed out which of my results coincided with his own.

Equation (2) may be put under the form

$$(-t+u+v+w)^2 = 4(vw+wu+uv);$$

* A very elegant demonstration of this theorem, by the late G. W. Hearn, will be found at p. 55 of his *Researches on Curves of the Second Order*. It is however only a particular case of a more general theorem, due it would seem to M. Chasles, for which see *Journal*, new series, vol. VI. p. 130, theorem (XIX.).

hence $vw + wu + uv = 0$ is the equation to the cone which has its vertex at the point (uvw) or A, and which envelopes the surface (2). In like manner we shall obtain the equations of the cones which have their vertices at the other angles of the tetrahedron $ABCD$, and which envelope the surface (2); and collecting the whole we have

$$\left.\begin{aligned} vw + wu + uv &= 0 \\ wt + tv + vw &= 0 \\ tu + uw + wt &= 0 \\ uv + vt + tu &= 0 \end{aligned}\right\} \text{..................(41).}$$

These I shall denominate the *circumscribed* cones of the given tetrahedron.

Again, if in (2) we put $t = 0$, $u = 0$, $v = 0$, and $w = 0$, in succession, we shall get

$$\left.\begin{aligned} u^2 + v^2 + w^2 - 2uv - 2uw - 2vw &= 0 \\ t^2 + v^2 + w^2 - 2tv - 2tw - 2vw &= 0 \\ t^2 + u^2 + w^2 - 2tu - 2tw - 2uw &= 0 \\ t^2 + u^2 + v^2 - 2tu - 2tv - 2uv &= 0 \end{aligned}\right\} \text{......(42);}$$

and these are evidently the equations to the cones whose vertices are at the angular points of the given tetrahedron, and whose directors are the conics in which the opposite faces of the tetrahedron intersect the surface. These may be called the *inscribed* cones of the given tetrahedron.

Taking the difference between the first two equations of (41), we get $(t - u)(v + w) = 0$; now $v + w = 0$ is the equation of a common tangent plane to the two cones denoted by the two equations; hence the cones *intersect* only in the plane $t - u = 0$: in like manner it may be shewn that the planes of the other conics in which the cones (41) intersect, two and two, are those denoted by (10)...(14); hence

XI. *The circumscribed cones intersect each other, two and two, in six planes which pass through one point.*

Again, taking the difference of the first two equations of (42), we get

$$\{t - u\}\{t + u - 2(v + w)\} = 0,$$

and hence the two cones denoted by the first two equations of (42) intersect in the two planes $t - u = 0$ and $t + u = 2(v + w)$. In a similar manner we shall obtain the equations to the other planes in which the inscribed cones intersect two and two; there will be twelve such planes altogether—six of

them coincide with those denoted by (9)...(14), and the equations to the other six are as follows:

$$\left.\begin{aligned} t + u &= 2(v + w) \\ t + v &= 2(u + w) \\ t + w &= 2(u + v) \\ v + w &= 2(t + u) \\ u + w &= 2(t + v) \\ u + v &= 2(t + w) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (43).$$

Now the first and fourth of these equations are satisfied by $t + u = v + w = 0$; the second and fifth by $t + v = u + w = 0$; and the third and sixth by $t + w = u + v = 0$. Hence, (15...20) and (III.),

XII. *The inscribed cones intersect each other, two and two, in twelve planes—which consist of a system of six planes that intersect in a point, and of a system of six planes, the corresponding pairs of which intersect in three straight lines in the same plane, and these straight lines coincide with the intersections of the three pairs of tangent planes to the surface* (2) *that pass through the opposite edges of the tetrahedron.*

It is evident from a comparison of (2) and (37), that if we accent t, u, v, and w in all the equations from (41) inclusive, we shall get the equations to the inscribed and circumscribed cones of the second tetrahedron $A'B'C'D'$, and their mutual intersections. Now, (35), $t' - u' = u - t$, &c., $t' + u' = v + w$, &c. identically; hence the following theorem is evident:

XIII. *The* $\left\{\begin{array}{l}\textit{inscribed}\\ \textit{circumscribed}\end{array}\right\}$ *cones of the second tetrahedron intersect each, two and two, in the same planes as the* $\left\{\begin{array}{l}\textit{inscribed}\\ \textit{circumscribed}\end{array}\right\}$ *cones of the given tetrahedron.*

Again, considering the circumscribed cones of both tetrahedra, each pair of corresponding cones will intersect in the plane $t + u + v + w = 0$; besides which there are *second* planes of intersection which are different for each pair of cones; their equations are,

$$\left.\begin{aligned} -3t + u + v + w &= 0 \\ -3u + t + v + w &= 0 \\ -3v + t + u + w &= 0 \\ -3w + t + u + v &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (44);$$

and each of these passes through the point O, or $t = u = v = w$. Hence

XIV. *The corresponding circumscribed cones of the two tetrahedra intersect each other in five planes—which consist of the single plane that passes through the four straight lines in which the corresponding faces of the two tetrahedra intersect; and of a system of four planes which pass respectively through these lines and which intersect in the point O.*

The corresponding inscribed cones will be found to intersect each other in the preceding five planes; so that

XV. *The corresponding inscribed cones of the two tetrahedra intersect each other in five planes which coincide with those in which the corresponding circumscribed cones intersect each other.*

Besides the faces of the given tetrahedron, the inscribed cones of the given tetrahedron intersect the corresponding circumscribed cones of the second tetrahedron in the planes

$$\left.\begin{aligned} \tfrac{3}{2}t + u + v + w &= 0 \\ \tfrac{3}{2}u + t + v + w &= 0 \\ \tfrac{3}{2}v + t + u + w &= 0 \\ \tfrac{3}{2}w + t + u + v &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (45).$$

Also, besides the faces of the second tetrahedron, the circumscribed cones of the given tetrahedron intersect the corresponding inscribed cones of the second tetrahedron in the planes

$$\left.\begin{aligned} \tfrac{3}{5}t + u + v + w &= 0 \\ \tfrac{3}{5}u + t + v + w &= 0 \\ \tfrac{3}{5}v + t + u + w &= 0 \\ \tfrac{3}{5}w + t + u + v &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (46).$$

Hence, including the faces of the two tetrahedra in which these cones also intersect, we may say that

XVI. *The sixteen planes in which the inscribed cones of each tetrahedron intersect the corresponding circumscribed cones of the other, pass through four straight lines in one plane, four planes passing through each line.*

Moreover, besides the faces of the given tetrahedron, the inscribed cones of the same intersect the surface (2) in four planes whose equations are

$$\left.\begin{aligned} -\tfrac{1}{2}t + u + v + w &= 0 \\ -\tfrac{1}{2}u + t + v + w &= 0 \\ -\tfrac{1}{2}v + t + u + w &= 0 \\ -\tfrac{1}{2}w + t + u + v &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (47).$$

Also, besides the faces of the second tetrahedron, the inscribed cones of the same intersect the surface (2) in the four planes

$$\left.\begin{aligned} 7t + u + v + w &= 0 \\ 7u + t + v + w &= 0 \\ 7v + t + u + w &= 0 \\ 7w + t + u + v &= 0 \end{aligned}\right\} \dots\dots\dots\dots\dots (48).$$

Hence, recollecting that the former and latter cones also intersect the given surface in the faces of the first and second tetrahedra respectively, we may say that

XVII. *The sixteen planes in which the inscribed cones of the two tetrahedra intersect the surface* (2) *pass, four and four, through four straight lines that are in one plane.*

We have thus a great number of planes, (see 44, 45, 46, 47, 48), which pass through the four lines (tt'), (uu'), (vv'), (ww'), in which the corresponding faces of the two tetrahedra intersect.

Several harmonic relations between the faces of the tetrahedra and the planes just referred to, might easily be deduced, but I shall omit them.

From (44) inclusive, I have only written down the equations to the planes of intersection in order to save space. The investigations are extremely easy, and I shall, as an example, merely investigate the equations for (xv.).

The equation to the inscribed cone of the second tetrahedron having its vertex at the point A' is of course

$$u'^2 + v'^2 + w'^2 - 2u'v' - 2u'w' - 2v'w' = 0,$$

which, when expressed in terms of t, u, v, and w, becomes

$$-3t^2 + 5u^2 + 5v^2 + 5w^2 - 2tu - 2tv - 2tw - 6uv - 6uw - 6vw = 0\,;$$

from this equation deduct four times

$$u^2 + v^2 + w^2 - 2uv - 2uw - 2vw = 0,$$

(the equation to the corresponding inscribed cone of the given tetrahedron), and we get

$$-3t^2 - 2t(u + v + w) + (u + v + w)^2 = 0,$$

or
$$(t + u + v + w)(-3t + u + v + w) = 0\,;$$

so that the two cones intersect in two plane curves situated in the planes $t + u + v + w = 0$ and $-3t + u + v + w = 0$.

It may also be observed, that having obtained (45) and (47), we shall get (46) and (48) by accenting the letters in (45) and (47), and then expressing the resulting equations in terms of t, u, v, and w by means of (35).

Putting $w=0$ in the first equation of (47), we have $t=2(u+v)$, and these values of t and w being substituted in (2), there results $(u-v)^2=0$, or $u-v=0$; hence the straight line in which the face w and the first plane of (47) intersect, touches the given surface. Similarly, it may be shewn that any of the planes (47) and a *non-corresponding* face of the given tetrahedron intersect in a straight line which touches the surface (2). Also, since the equations (48) are what (47) become when t, u, v, and w are accented, it follows that the same is true of the planes (48) and the faces of the second tetrahedron. Hence

XVIII. *The twelve straight lines in which the four planes* (47) *intersect the non-corresponding faces of the given tetrahedron, and the twelve straight lines in which the planes* (48) *intersect the non-corresponding faces of the second tetrahedron, touch the given surface* (2 *or* 37).

It will be observed that the faces of the given tetrahedron and the planes (47) form the faces of a duodecangular octahedron whose edges touch the surface (2) of the second degree; but as the consideration of this subject here might be too great a digression, I shall postpone it to the end of this paper.

The equation to any surface of the second degree circumscribed about the given tetrahedron is evidently

$$ftu+gtv+htw+f'vw+g'uw+h'uv=0.$$

If this also passes through the points A', B', C', and D', we must, (39), have

$$-f-g-h+f'+g'+h'=0,$$
$$-f+g+h+f'-g'-h'=0,$$
$$f-g+h-f'+g'-h'=0,$$

and

$$f+g-h-f'-g'+h'=0.$$

Now the last three equations give $f=f'$, $g=g'$, and $h=h'$, and these satisfying the first equation, we infer that

XIX. *Any surface of the second degree passing through seven of the angular points of the two tetrahedra will pass through the eighth.**

* Since the angular points of the two tetrahedra form the angular points of a hexahedron (whose faces are denoted by

$$t+u=0,\quad v+w=0,$$
$$t+v=0,\quad w+u=0,$$
$$t+w=0,\quad u+v=0),$$

The general equation to such a surface is

$$f(tu+vw)+g(tv+uw)+h(tw+uv)=0 \ldots\ldots (49),$$

f, g, and h being arbitrary constants.

The last equation may be presented in a neat form in terms of p, q, r, and s. By (35) we have

$$tu+vw=\tfrac{1}{2}(p^2+q^2-r^2-s^2), \text{ \&c.},$$

hence (49) becomes

$$(f+g+h)p^2+(f-g-h)q^2+(-f+g-h)r^2+(-f-g+h)s^2=0.$$

Put $k=f+g+h$, $l=f-g-h$, $m=-f+g-h$, and $n=-f-g+h$, so that $k+l+m+n=0$. Hence the equation to surfaces of the second degree circumscribed about the given and second tetrahedra may be written in this form:

$$\left.\begin{aligned} kp^2+lq^2+mr^2+ns^2&=0\\ \text{where}\qquad k+l+m+n&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots (50).$$

XX. *Any surface of the second degree touching seven of the faces of the two tetrahedra, will touch the eighth face.*

This theorem may be established as follows:

In the *Mathematician*, vol. III. p. 278, I have shewn that the general equation to surfaces of the second degree touching the faces t, u, v, w of the given tetrahedron is

$$l(at+bu-cv-ew)^2+m(at-bu+cv-ew)^2+n(at-bu-cv+ew)^2$$
$$=(at+bu+cv+ew)^2,$$

where $\qquad l^{-1}+m^{-1}+n^{-1}=1.$

Substitute $-k^{-1}l$, $-k^{-1}m$, and $-k^{-1}n$, for l, m, and n, and these equations become

$$k(at+bu+cv+ew)^2+l(at+bu-cv-ew)^2+m(at-bu+cv-ew)^2$$
$$+n(at-bu-cv+ew)^2=0 \ldots (1').*$$

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it follows that (XIX.) is only a particular case of theorem (VIII.) in my second memoir "On the Theorems in Space analogous to those of Pascal and Brianchon in a Plane," (*Journal*, new series, vol. V. p. 65). But indeed the latter theorem itself is only a particular case of the following: *Every surface of the second degree passing through seven of the eight points of intersection of three surfaces of the second degree, will necessarily pass through the eighth.*

I may as well observe here that, since the faces of the two tetrahedra form those of an octahedron, the next theorem (XX.) in the text is only a particular case of theorem (IX.) (*ib.* p. 66), and this again is only a particular case of the following: *Every surface of the second degree touching seven of the eight common tangent planes to three surfaces of the second degree, will necessarily touch the eighth.*

* If we change the sign of the arbitrary constant a in this equation, we shall get the more symmetrical form

where $$k^{-1}+l^{-1}+m^{-1}+n^{-1}=0 \text{ ....}(2').$$

Put
$$at+bu+cv+ew=2P,$$
$$at+bu-cv-ew=2Q,$$
$$at-bu+cv-ew=2R,$$
$$at-bu-cv+ew=2S;$$

also put
$$\left.\begin{aligned} -a^{-1}+b^{-1}+c^{-1}+e^{-1}&=2\alpha \\ a^{-1}-b^{-1}+c^{-1}+e^{-1}&=2\beta \\ a^{-1}+b^{-1}-c^{-1}+e^{-1}&=2\gamma \\ a^{-1}+b^{-1}+c^{-1}-e^{-1}&=2\delta \end{aligned}\right\} \text{ ....}(3').$$

We thus readily find

$$\left.\begin{aligned} -2t' &= -\alpha P+\beta Q+\gamma R+\delta S \\ 2u' &= \beta P-\alpha Q+\delta R+\gamma S \\ 2v' &= \gamma P+\delta Q-\alpha R+\beta S \\ 2w' &= \delta P+\gamma Q+\beta R-\alpha S \end{aligned}\right\} \text{ ....}(4');$$

and the equation (1′) to the tangent surface becomes

$$kP^2+lQ^2+mR^2+nS^2=0 \text{ ....}(5').$$

Hence, observing (4′) and the foot-note at p. 110, we see that if $t'=0$, $u'=0$, $v'=0$, and $w'=0$, denote tangent planes to (1′) or (5′), we must have

$$\left.\begin{aligned} k^{-1}\alpha^2+l^{-1}\beta^2+m^{-1}\gamma^2+n^{-1}\delta^2&=0 \\ k^{-1}\beta^2+l^{-1}\alpha^2+m^{-1}\delta^2+n^{-1}\gamma^2&=0 \\ k^{-1}\gamma^2+l^{-1}\delta^2+m^{-1}\alpha^2+n^{-1}\beta^2&=0 \\ k^{-1}\delta^2+l^{-1}\gamma^2+m^{-1}\beta^2+n^{-1}\alpha^2&=0 \end{aligned}\right\} \text{ ....}(6').$$

Now if we add these four equations, we get

$$(k^{-1}+l^{-1}+m^{-1}+n^{-1})(\alpha^2+\beta^2+\gamma^2+\delta^2),$$

which $=0$, by (2′). Hence three of the equations (6′) imply the fourth, and theorem (xx.) is therefore established.

$$k(-at+bu+cv+ew)^2+l(at-bu+cv+ew)^2+m(at+bu-cv+ew)^2$$
$$+n(at+bu+cv-ew)^2=0,$$
where $$k^{-1}+l^{-1}+m^{-1}+n^{-1}=0.$$

Another form (also symmetrical) of the general equation to surfaces of the second degree touching the faces of a tetrahedron is made use of at vol. v. p. 61 of this *Journal;* and the demonstrations of the whole will be found in the *Mathematician*, vol. II. p. 261, and vol. III. pp. 278-9.

Add the first two equations of (6′), recollecting that $k^{-1} + l^{-1} = -(m^{-1} + n^{-1})$, therefore

$$(k^{-1} + l^{-1})(\alpha^2 + \beta^2 - \gamma^2 - \delta^2) = 0;$$

therefore either $k^{-1} + l^{-1} = 0$, or $\alpha^2 + \beta^2 = \gamma^2 + \delta^2$:

but, (3′), $\alpha^2 + \beta^2 = \gamma^2 + \delta^2$ is equivalent to $ab = ce$,

therefore either $k^{-1} + l^{-1} = 0$, or $ab = ce$.

Similarly, $k^{-1} + m^{-1} = 0$, or $ac = be$,

and $k^{-1} + n^{-1} = 0$, or $ae = bc$.

Now, since the equations $k^{-1} + l^{-1} = 0$, $k^{-1} + m^{-1} = 0$, and $k^{-1} + n^{-1} = 0$, cannot exist simultaneously (2′), we shall have three cases to consider:

Firstly, $ab = ce$, $ac = be$, and $ac = bc$.

Secondly, such as

$$k^{-1} + l^{-1} = 0, \quad ac = be, \quad \text{and } ae = bc.$$

Thirdly, such as

$$k^{-1} + l^{-1} = 0, \quad k^{-1} + m^{-1} = 0, \quad \text{and } ae = bc.$$

Taking the first of these, we must have

either $a = b = c = e$,

or, such as $a = b = -c = -e$.

Substituting the former of these in (1′), we find the required equation to be

$$k(t + u + v + w)^2 + l(t + u - v - w)^2 + m(t - u + v - w)^2 + n(t - u - v + w)^2 = 0 \ldots\ldots (7').$$

If we were to take $a = b = -c = -e$ in (1′), and then interchange k and l, and m and n, (which does not affect (2′)), we should also get (7′).

Secondly, let $k^{-1} + l^{-1} = 0$, $ac = be$, and $ae = bc$; hence, (2′), $l = -k$ and $n = -m$. Also either $a = b$ and $c = e$; or $a = -b$ and $c = -e$. Taking $l = -k$, $n = -m$, $b = a$, and $e = c$, in (1′), we get

$$k(t + u)(v + w) + m(t - u)(v - w) = 0;$$

also taking $l = -k$, $n = -m$, $b = -a$, and $c = -e$ in (1′), we get

$$k(t - u)(v - w) + m(t + u)(v + w) = 0,$$

and this coincides with the former when k and m are interchanged; but the former equation is what (7′) becomes when $l = -k$ and $n = -m$, so that both these equations may be rejected as being only cases of (7′).

Lastly, let $k^{-1}+l^{-1}=0$, $k^{-1}+m^{-1}=0$, and $ae=bc$; hence (2′), $k=-l=-m=n$; and (1′) is reduced to

$$aetw+bcuv=0:$$

but since $ae=bc$, this becomes $tw+uv=0$, and this equation may also be rejected, seeing that it results from putting $k=-l=-m=n$ in (7′).

From the preceding discussion it follows that the general equation to surfaces of the second degree touching the faces of the given and second tetrahedra may, (7′), be written

$$k(t+u+v+w)^2+l(-t-u+v+w)^2+m(-t+u-v+w)^2$$
$$+n(-t+u+v-w)^2=0 \ldots (51),$$

where k, l, m, n are arbitrary constants, subject however to the condition

$$k^{-1}+l^{-1}+m^{-1}+n^{-1}=0 \ldots\ldots\ldots\ldots\ldots\ldots (52).$$

The equation (51) may also be written

$$kp^2+lq^2+mr^2+ns^2=0 \ldots\ldots\ldots\ldots\ldots\ldots (53)^*,$$

also of course with the condition (52).

Since (2) may be written in the form

$$(t-u)^2+(v-w)^2-2(t+u)(v+w)=0,$$

we see that the equation to any other surface of the second degree touching the edges of the given tetrahedron is

$$(at-bu)^2+(cv-ew)^2-2(at+bu)(cv+ew)=0 \ldots (\alpha).$$

Let this touch the edge $C'D'$ or $(t'u')$ of the second tetrahedron; then, since $t'=u'=0$ are equivalent to $t=u$, and $w=-v$, we must have, (α),

$$(a-b)^2u^2+(c+e)^2v^2-2(a+b)(c-e)vw=0,$$

---

* The faces of the second and given tetrahedra, when expressed in terms of p, q, r, and s, are, (35), denoted as follows:

$$\left.\begin{array}{r} p+q+r+s=0 \\ -p-q+r+s=0 \\ -p+q-r+s=0 \\ -p+q+r-s=0 \end{array}\right\} \text{ and } \left\{\begin{array}{l} -p+q+r+s=0 \\ p-q+r+s=0 \\ p+q-r+s=0 \\ p+q+r-s=0. \end{array}\right.$$

Hence (*Journal*, new series, vol. VII. p. 256, equations (5)), the faces of the two tetrahedra are the faces of a diagrammatic octahedron. Consequently the above investigation of (51), that is, of (53), (with the condition (52)) does in fact establish equations (9) (*ibid.* p. 258) in my memoir "On certain Systems in Space analogous to the Complete Tetragon and Complete Quadrilateral;" and hence also equations (54) in my first memoir "On the Theorems in Space analogous to those of Pascal and Brianchon in a Plane," (*ibid.* vol. IV. p. 37). It is on this account chiefly that the above investigation has been inserted.

a complete square; this requires that

$$(a-b)(c+e) = \pm(a+b)(c-e);$$

which gives $ac = be$ or $ae = bc$.

Proceeding in this way, we find the conditions required in order that the surface (α) should touch the various edges of the second tetrahedron to be as follows:

If the surface (α) touch $A'B'$ or $C'D'$, then must

either $ac = be$ or $ae = bc$ .... (β),

if $A'C'$ or $B'D'$, either $ab = ce$ or $ae = bc$ .... (γ),

if $A'D'$ or $B'C'$, either $ab = ce$ or $ac = be$ .... (δ).

Hence, if the surface (α) touch any edge of the second tetrahedron, it must necessarily touch the opposite edge; also if the surface touch any edge, say $A'B'$, so that one of the conditions (β) is satisfied, then one of the conditions either of (γ) or of (δ) will be satisfied, so that the surface will touch other two opposite edges. Hence this theorem:

XXI. *Every surface of the second degree touching all the edges of the given tetrahedron, and also one of the edges of the second tetrahedron, will necessarily touch the opposite edge, and likewise one or other of the two remaining pairs of opposite edges.*

In order that the surface (α) should also touch all the edges of the second tetrahedron, we must satisfy (β), (γ), and (δ), and this may be effected in three ways:

either $ac = be$ and $ae = bc$ .... (ε),

or $ab = ce$ and $ae = bc$ ....(η),

or $ab = ce$ and $ac = be$ .... (ζ).

(ε) gives $b = \pm a$ and $e = \pm c$,

(η) gives $c = \pm a$ and $b = \pm e$,

(ζ) gives $e = \pm a$ and $c = \pm b$.

Substituting these values successively in (α), we readily find the resulting equations to be

$$a^2(t \mp u)^2 + c^2(v \mp w)^2 = 2ac(t \pm u)(v \pm w) \quad\text{......(54)},$$

$$a^2(t \mp v)^2 + e^2(w \mp u)^2 = 2ae(t \pm v)(w \pm u) \quad\text{......(55)},$$

$$a^2(t \mp w)^2 + b^2(u \mp v)^2 = 2ab(t \pm w)(u \pm v) \quad\text{......(56)},$$

where a, b, c, and e are arbitrary constants, and each equation, on account of the double signs, is equivalent to two.

Hence, every surface of the second degree touching all the edges of both tetrahedra, must have an equation of some

of the above forms. But each of the surfaces (54), (55), and (56), whatever may be the values of the constants, always touches one pair of edges of each tetrahedron at their points of intersection: thus (54), if the upper signs be taken, touches the edges CD and $A'B'$ at their point of intersection e, and the edges AB and $C'D'$ at their point of intersection E; while, if the lower signs be taken, (54) touches the edges CD and $C'D'$ at their point of intersection e_1, and the edges AB and $A'B'$ at their point of intersection E_1. Hence

XXII. *A surface of the second degree that touches all the edges of both tetrahedra must touch one pair of opposite edges of the given tetrahedron, and the corresponding pair of opposite edges of the second tetrahedron at the points of intersection of these edges; but it does not necessarily touch the other eight edges at their points of intersection.*

The two tetrahedra possess other properties of a similar kind, but I deem it advisable to omit them.

The equation $tu + vw = 0$ denotes a ruled hyperboloid, which being satisfied by $t = v = 0$, $t = w = 0$, $u = v = 0$, and $u = w = 0$, respectively, passes through two pairs of opposite edges of the given tetrahedron. Now $tu + vw = t'u' + v'w'$ identically, and hence the hyperboloid also passes through the corresponding two pairs of edges of the second tetrahedron. Hence

XXIII. *Through any two pairs of opposite edges of the given tetrahedron, and the corresponding two pairs of opposite edges of the second tetrahedron, a ruled hyperboloid may be described.*

There are evidently three such surfaces, and their equations are

$$\left.\begin{aligned} tu + vw &= 0 \\ tv + uw &= 0 \\ tw + uv &= 0 \end{aligned}\right\} \quad \text{..................(57);}$$

or, (35),

$$\left.\begin{aligned} p^2 + q^2 &= r^2 + s^2 \\ p^2 + r^2 &= q^2 + s^2 \\ p^2 + s^2 &= q^2 + r^2 \end{aligned}\right\} \quad \text{..................(58).}$$

By comparing (38) and (58), we see that the given surface touches the surfaces (58) in the planes $q = 0$, $r = 0$, and $s = 0$ respectively; or $t + u = v + w$, $t + v = u + w$, and $t + w = u + v$; and these pass through the lines in which the tangent planes (15) and (18), (16) and (19), and (17) and (20) intersect. Hence

XXIV. *Each of the hyperboloids just mentioned touches the given surface* (2) *in a plane. Also the plane of contact of any of these surfaces and the tangent planes to the given surface that pass through the third pairs of opposite edges intersect in a straight line.*

Equation (54) may be written in either of the forms

$$\{a(t \mp u) + c(v \mp w)\}^2 = 4ac(tv + uw),$$

or

$$\{a(t \mp u) - c(v \mp w)\}^2 = \pm\, 4ac(tw + uv);$$

and (55) and (56) may be thrown into similar forms. Hence

XXV. *Every surface of the second degree touching all the edges of both tetrahedra has necessarily plane contact with two of the three hyperboloids* (57).

In the *Mathematician*, vol. III., p. 278, I have shewn that the general equation to surfaces of the second degree touching the faces of the tetrahedron ($tuvw$), and which moreover are such that the straight lines joining the points of contact and the opposite angles shall intersect in a point, is

$$a^2t^2 + b^2u^2 + c^2v^2 + e^2w^2 - abtu - actv - aetw - bcuv - beuw - cevw = 0,$$

and the point through which the said lines pass is denoted by

$$at = bu = cv = ew;$$

if this point coincide with O or $t = u = v = w$ we must have $a = b = c = e$, and then the equation to the surface becomes

$$t^2 + u^2 + v^2 + w^2 - tu - tv - tw - uv - uw - vw = 0 \ldots (59).$$

Hence (59) is the equation to the surface which touches the faces of the given tetrahedron in the points in which they are intersected by the lines joining the corresponding angular points of the given and second tetrahedra. But since (59) still denotes the same surface when t', u', v', and w' are written for t, u, v, and w, it follows that the surface (59) also touches the faces of the second tetrahedron in the points in which these faces are intersected by the lines joining the corresponding angles of the two tetrahedra.

If we put $t = 0$ in (59), the resulting equation may be written

$$(-2u + v + w)^2 + 3(v - w)^2 = 0,$$

which requires $-2u + v + w = v - w = 0$, so that the tangent plane $t = 0$ meets the surface (59) in one point only.

XXVI. *An umbilical surface of the second degree can be described to touch the faces of the two tetrahedra in the points in which the said faces are intersected by the straight lines joining the corresponding angles of the two tetrahedra.*

Suppose every inscribed as well as every circumscribed cone of each tetrahedron to be limited by the corresponding face (of the same tetrahedron) which will thus be the *base* of the cone. A surface will be circumscribed about one of these cones when it passes through its vertex and the perimeter of its base; and will be inscribed in the same when it touches the base and the curved surface of the cone in a curve.

The equation (59) may be written

$$(-2t+u+v+w)^2+3(u^2+v^2+w^2-2uv-2uw-2vw)=0;$$

hence the inscribed cone, $u^2+v^2+w^2-2uv-2uw-2vw=0$, touches the surface (59) in the conic in which it is intersected by the plane $-2t+u+v+w=0$, and we already know, (XXVI.), that the base of the cone touches (59); hence the surface (59) is inscribed in this cone. Similarly, putting (59) under the form

$$t'^2+u'^2+v'^2+w'^2-t'u'-t'v'-t'w'-u'v'-u'w'-v'w'=0,$$

it may be shewn that the surface is inscribed in the inscribed cone of the second tetrahedron which has its vertex at the point A'. Hence

XXVII. ***An umbilical surface of the second degree may be inscribed in the eight inscribed cones of the two tetrahedra.***

Again, the surface whose equation is

$$tu+tv+tw+uv+uw+vw=0 \text{ ....(60)},$$

is circumscribed about all the circumscribed cones, for it is satisfied by $t=0$ and $uv+uw+vw=0$; by $u=0$ and $tv+tw+vw=0$, &c.; also, putting (60) under the form

$$t'u'+t'v'+t'w'+u'v'+u'w'+v'w'=0,$$

we see that it is satisfied by $t'=0$ and $u'v'+u'w'+v'w'=0$, &c.; also, since (60) is circumscribed about both tetrahedra, (49), it passes through the vertices of all the circumscribed cones. Hence

XXVIII. ***An umbilical surface of the second degree may be circumscribed about all the eight circumscribed cones of the two tetrahedra.***

The equations (59) and (60) may also be written

$$q^2+r^2+s^2=\tfrac{1}{3}p^2 \text{.....................(61)},$$

and

$$q^2+r^2+s^2=3p^2 \text{.....................(62)},$$

and hence both surfaces are umbilical, as has been asserted. Again,

XXIX. *A ruled hyperboloid may be described so as to touch two corresponding circumscribed cones of the two tetrahedra along the perimeters of their bases.*

The equations to these four surfaces are

$$\left.\begin{aligned} t^2 + uv + uw + vw &= 0 \\ u^2 + tv + tw + vw &= 0 \\ v^2 + tu + tw + uw &= 0 \\ w^2 + tu + tv + uv &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots(63).$$

For the surface $t^2 + uv + uw + vw = 0$ touches the cone $uv + uw + vw = 0$, in the plane $t = 0$; and since $t^2 + uv + uw + vw = t'^2 + u'v' + u'w' + v'w'$, identically, the same surface touches the cone $u'v' + u'w' + v'w' = 0$ in the plane $t' = 0$. Also $t^2 + uv + uw + vw = (t + u)(t - u) + (u + v)(u + w)$, hence $t^2 + uv + uw + vw = 0$ denotes a ruled hyperboloid.

We have already seen that the two tetrahedra $ABCD$ and $A'B'C'D'$ are copolar, that is, the four lines joining the corresponding angles A and A', B and B', C and C', and D and D' intersect in a point—the *pole* of the tetrahedra. There are, however, altogether *four* ways in which the tetrahedra are copolar, the angular points of the tetrahedron $OO'O''O'''$ coinciding with their poles. If we write the corresponding angular points of the tetrahedra in the same order, so that if, for instance, $ABCD$ and $B'A'D'C'$ be the tetrahedra, the corresponding angles are A and B', B and A', C and D', and D and C'; then the four ways in which the tetrahedra are copolar are as under:—

1. The tetrahedra $ABCD$ and $A'B'C'D'$ have O for pole; their edges AB and $C'D'$, AC and $B'D'$, AD and $B'C'$, CD and $A'B'$, BD and $A'C'$, BC and $A'D'$, intersect in the points E, F, G, e, f, g, respectively; and the surface of the second degree whose equation is

$$q^2 + r^2 + s^2 = p^2 \ldots\ldots\ldots\ldots\ldots\ldots(64),$$

touches the edges of both tetrahedra at these points.

2. The tetrahedra $ABCD$ and $B'A'D'C'$ have O' for pole; their edges AB and $D'C'$, AC and $A'C'$, AD and $A'D'$, CD and $B'A'$, BD and $B'D'$, BC and $B'C'$, intersect in the points E, F_1, G_1, e, f_1, g_1; and the surface of the second degree whose equation is

$$r^2 + s^2 + p^2 = q^2 \ldots\ldots\ldots\ldots\ldots\ldots(65),$$

touches the edges of both tetrahedra at these points.

3. The tetrahedra $ABCD$ and $C'D'A'B'$ have O'' for pole; their edges AB and $A'B'$, AC and $D'B'$, AD and $A'D'$, CD and $C'D'$, BD and $A'C'$, BC and $B'C'$, intersect in the points E_1, F, G_1, e_1, f, g_1, respectively; and the surface

$$s^2 + p^2 + q^2 = r^2 \text{ .... } (66),$$

touches the edges of both tetrahedra at these points.

4. The tetrahedra $ABCD$ and $D'C'B'A'$ have O''' for pole; their edges AB and $A'B'$, AC and $A'C'$, AD and $B'C'$, CD and $C'D'$, BD and $B'D'$, BC and $A'D'$, intersect in the points E_1, F_1, G, e_1, f_1, g, respectively; and the surface

$$p^2 + q^2 + r^2 = s^2 \text{ .... } (67),$$

touches the edges of both tetrahedra at these points.

Moreover the two tetrahedra $ABCD$ and $OO'O''O'''$ are copolar in four ways.

1'. The tetrahedra $ABCD$ and $OO'O''O'''$ have A' for pole; their edges AB and $O''O'''$, AC and $O'O'''$, AD and $O'O''$, CD and OO', BD and OO'', BC and OO''', intersect in the points E_1, F_1, G_1, e, f, g, respectively; and the surface

$$u'^2 + v'^2 + w'^2 = t'^2 \text{ .... } (68),$$

touches the edges of both tetrahedra at these points.

2'. The tetrahedra $ABCD$ and $O'OO'''O''$ have B' for pole; their edges AB and $O''O'''$, AC and OO'', AD and OO''', CD and OO', BD and $O'O'''$, BC and $O'O''$, intersect in the points E_1, F, G, e, f_1, g_1, respectively; and the surface

$$v'^2 + w'^2 + t'^2 = u'^2 \text{ .... } (69),$$

touches the edges of both tetrahedra at these points.

3'. The tetrahedra $ABCD$ and $O''O'''OO'$ have C' for pole; their edges AB and OO', AC and $O'O'''$, AD and OO''', CD and $O''O'''$, BD and OO'', BC and $O'O''$, intersect in the points E, F_1, G, e_1, f, g_1, respectively; and the surface

$$w'^2 + t'^2 + u'^2 = v'^2 \text{ .... } (70),$$

touches the edges of both tetrahedra at these points.

4'. The tetrahedra $ABCD$ and $O'''O''O'O$ have D' for pole; their edges AB and OO', AC and OO'', AD and $O'O''$, CD and $O''O'''$, BD and $O'O'''$, BC and OO''', intersect in the points E, F, G_1, e_1, f_1, g, respectively; and the surface

$$t'^2 + u'^2 + v'^2 = w'^2 \text{ .... } (71),$$

touches the edges of both tetrahedra at these points.

Lastly, the two tetrahedra $A'B'C'D'$ and $OO'O''O'''$ are copolar in four different ways.

1″. The tetrahedra $A'B'C'D'$ and $OO'O''O'''$ have A for pole; their edges $A'B'$ and $O''O'''$, $A'C'$ and $O'O'''$, $A'D'$ and $O'O''$, $C'D'$ and OO', $B'D'$ and OO'', $B'C'$ and OO''', intersect in the points E_1, F_1, G_1, E, F, G, respectively; and the surface

$$u^2 + v^2 + w^2 = t^2 \quad\text{..................} \quad (72),$$

touches the edges of both tetrahedra at these points.

2″. The tetrahedra $A'B'C'D'$ and $O'OO'''O''$ have B for pole; their edges $A'B'$ and $O''O'''$, $A'C'$ and OO'', $A'D'$ and OO''', $C'D'$ and OO', $B'D'$ and $O'O'''$, $B'C'$ and $O'O''$, intersect in the points E_1, f, g, E, f_1, g_1, respectively; and the surface

$$v^2 + w^2 + t^2 = u^2 \quad\text{..................} \quad (73),$$

touches the edges of both tetrahedra at these points.

3″. The tetrahedra $A'B'C'D'$ and $O''O'''OO'$ have C for pole; their edges $A'B'$ and OO', $A'C'$ and $O'O'''$, $A'D'$ and OO''', $C'D'$ and $O''O'''$, $B'D'$ and OO'', $B'C'$ and $O'O''$, intersect in the points e, F_1, g, e_1, F, g_1, respectively; and the surface

$$w^2 + t^2 + u^2 = v^2 \quad\text{.....................} \quad (74),$$

touches the edges of both tetrahedra at these points.

4″. The tetrahedra $A'B'C'D'$ and $O'''O''O'O$ have D for pole; their edges $A'B'$ and OO', $A'C'$ and OO'', $A'D'$ and $O'O''$, $C'D'$ and $O''O'''$, $B'D'$ and $O'O'''$, $B'C'$ and OO''', intersect in the points e, f, G_1, e_1, f_1, G, respectively; and the surface

$$t^2 + u^2 + v^2 = w^2 \quad\text{.....................} \quad (75),$$

touches the edges of both tetrahedra at these points.

Hence the following theorem:

XXX. *Any two of the three tetrahedra $ABCD$, $A'B'C'D'$ and $OO'O''O'''$ are copolar in four different ways; and the poles of any two coincide with the angular points of the third tetrahedron. Also through the six points in which the edges of any two of these copolar tetrahedra intersect, an umbilical surface of the second degree may be described so as to touch all the twelve edges.*

There are, as we have already seen, twelve such surfaces, and it may be noted that those eight which touch the edges of the tetrahedron $ABCD$ are the eight umbilical surfaces mentioned in (II.).

Each of the following sets of planes forms a harmonic system: v, w, $v + w$, $v - w$; $v + w$, $t + u$, $2p = (v + w) + (t + u)$, $2q = (v + w) - (t + u)$; $u - t$, $v - w$, $2s = (u - t) + (v - w)$,

$2r = (u - t) - (v - w)$; $v + w$, $u - t$, $2t' = (v + w) + (u - t)$, $2u' = (v + w) - (u - t)$; &c., &c.; hence the following theorem is evident.

XXXI. *Some two of the twelve planes,* (3...8) *and* (15...20), *pass through each of the eighteen edges of the three tetrahedra* $ABCD$, $A'B'C'D'$, *and* $OO'O''O'''$; *also the two planes and the two faces that pass through any edge form a harmonic system.*

It is evident from the symmetry of the equations (35) that any property which the given and second tetrahedra may possess, will also be true (with suitable modifications) of *any* two of the three tetrahedra. Thus, on substituting $t'u'v'w'$ instead of $pqrs$ in (50) or (53), we shall have the general equation to surfaces of the second degree circumscribed about, or inscribed in, the two tetrahedra $ABCD$ and $OO'O''O'''$; and if $tuvw$ be written for $pqrs$ in the same, we shall get the equations for the tetrahedra $A'B'C'D'$ and $OO'O''O'''$.

From this and the equations (64...75) it appears that

XXXII. *The angular points of any one of the three tetrahedra* $ABCD$, $A'B'C'D'$, *and* $OO'O''O'''$ *are the poles of the opposite faces (of the same tetrahedron), with respect to any of the following surfaces of the second degree:*

1. *Any surface circumscribed about the other two tetrahedra;*
2. *Any surface inscribed in the same; and*
3. *Each of the four surfaces touching the edges of the other two tetrahedra at the points of intersection of their edges.*

Again, it is obvious that the equations to the angles of the three tetrahedra, namely (39, 40), and the equations

$$\left.\begin{array}{l} u = v = w = 0 \\ v = w = t = 0 \\ w = t = u = 0 \\ t = u = v = 0 \end{array}\right\}$$

coincide in form with equations (3) and (4) in my memoir "On certain Systems in Space analogous to the Complete Tetragon and Complete Quadrilateral," (*Journal*, new series, vol. VII. p. 254); also it is equally evident that the equations to the faces of the three tetrahedra coincide in form with the equations at the foot of p. 257 (*ibid.*). Hence

XXXIII. *The angles of any two of the tetrahedra* $ABCD$, $A'B'C'D'$, *and* $OO'O''O'''$ *form the angles of a complete syngrammatic octangle whose syngrammatic points are the angles of the other tetrahedron; and the faces of any two form the faces of a complete*

*diagrammatic octahedron whose diagrammatic planes are the faces of the remaining tetrahedron.**

I now come to consider some formulas and properties relative to the centres of gravity of the tetrahedra, and the centres of various surfaces and curves of the second degree.

Let a, b, c, and e be such constants that

$$at + bu + cv + ew = 1 \quad\ldots\ldots\ldots\ldots\ldots (76),$$

identically. I shall occasionally refer to this as the *equation of identity.*

Let

$$\left.\begin{aligned} 2a' &= -a + b + c + e \\ 2b' &= a - b + c + e \\ 2c' &= a + b - c + e \\ 2e' &= a + b + c - e \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (77),$$

also

$$\left.\begin{aligned} 2a'' &= a + b + c + e \\ 2b'' &= -a - b + c + e \\ 2c'' &= -a + b - c + e \\ 2e'' &= -a + b + c - e \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (78).$$

Hence, (35), the equation of identity is equivalent to either of the following equations :

$$a't' + b'u' + c'v' + e'w' = 1 \ldots\ldots\ldots\ldots\ldots (79),$$

$$a''p + b''q + c''r + e''s = 1 \ldots\ldots\ldots\ldots\ldots (80).$$

When $u = v = w = 0$, we have $at = 1$ identically, (76); hence the plane, $at = 1$, passes through the point A, and it is parallel to the face BCD: but t is proportional to the perpendicular from A on BCD; consequently the equation to the plane, parallel to the face BCD and at a distance from it equal to one-fourth of this perpendicular, is $at = \frac{1}{4}$; and this plane of course contains the centre of gravity of the tetrahedron $ABCD$. Similarly, the planes $bu = \frac{1}{4}$, $cv = \frac{1}{4}$, and $ew = \frac{1}{4}$, all contain the centre of gravity; hence this centre is denoted by

$$at = bu = cv = ew = \tfrac{1}{4} \ldots\ldots\ldots\ldots\ldots (81).\dagger$$

* For the definitions of 'complete syngrammatic octangle' and 'complete diagrammatic octahedron,' see the memoir just referred to. (*Journal*, new series, vol. VII. pp. 255 and 257.)

† The neat equations (81) to the centre of gravity of a tetrahedron were discovered by the late Mr. Hearn and myself independently, the demonstration given above being however Mr. Hearn's. Equations (82) and (83) are due to myself.

Let K denote the centre of gravity of the tetrahedron $ABCD$; K_1, K_2, K_3, K_4 those of its faces BCD, ACD, ABD, ABC, respectively; and L, M, N, l, m, n, the middle points of the edges AB, AC, AD, CD, BD, BC, respectively. Also let these letters accented apply to the tetrahedron $ABCD$; and the same letters with two accents apply to the tetrahedron $OO'O''O'''$.

By (81) the equations to the line AK which intersects the face BCD in K_1 are $bu = cv = ew$; hence, when $t = 0$, we have, (76), $bu = cv = ew = \frac{1}{3}$. The centres of gravity of the faces of the tetrahedron $ABCD$ are therefore denoted as follows:

$$\left.\begin{array}{lll} K_1, & t = 0, & bu = cv = ew = \frac{1}{3} \\ K_2, & u = 0, & at = cv = ew = \frac{1}{3} \\ K_3, & v = 0, & at = bu = ew = \frac{1}{3} \\ K_4, & w = 0, & at = bu = cv = \frac{1}{3} \end{array}\right\} \text{.........} \ (82).$$

Again, from these equations we see that the plane, $at = bu$, contains the points K_3K_4, and it passes through the edge CD; consequently this plane bisects the edge AB, and therefore the middle point (L) of AB is denoted by $v = w = 0$ and $at = bu = \frac{1}{2}$, by (76). Hence the middle points of the edges of the tetrahedron $ABCD$ are denoted as follows:

$$\left.\begin{array}{lll} L, & v = w = 0, & at = bu = \frac{1}{2} \\ M, & w = u = 0, & at = cv = \frac{1}{2} \\ N, & u = v = 0, & at = ew = \frac{1}{2} \\ \mathrm{l}, & t = u = 0, & cv = ew = \frac{1}{2} \\ \mathrm{m}, & t = v = 0, & ew = bu = \frac{1}{2} \\ \mathrm{n}, & t = w = 0, & bu = cv = \frac{1}{2} \end{array}\right\} \text{.........} (83)^*.$$

If all the letters in (81), (82) and (83) be accented, it is evident that the equations will then be adapted to the second tetrahedron $A'B'C'D'$. And if a'', b'', c'', e'' be written for a, b, c, e, and p, q, r, s for t, u, v, w, they will apply to the

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* In like manner if $t=0$, $u=0$, $v=0$, be the equations to the sides of a plane triangle, and

$$at+bu+cv=1,$$

be the equation of identity, then the equations to the centre of gravity of the triangle are

$$at=bu=cv=\tfrac{1}{3};$$

and those to the middle points of the sides are

$$t=0, \quad bu=cv=\tfrac{1}{2};$$

$$u=0, \quad cv=at=\tfrac{1}{2};$$

and $$v=0, \quad at=bu=\tfrac{1}{2}, \text{ respectively.}$$

third tetrahedron $OO'O''O'''$. It is unnecessary therefore to write down the formulas for these two tetrahedra.

The preceding equations (81), (82), and (83) may be viewed somewhat more generally. If instead of $at+bu+cv+ew$ being $=1$ identically (so that $at+bu+cv+ew=0$ is the equation of the plane at infinity), we assume

$$at + bu + cv + ew = 0$$

to be the equation to any plane; then (omitting '$=\frac{1}{2}$') (83) will denote the harmonic conjugates of the points in which the plane cuts the edges of the tetrahedron $(tuvw)$, the harmonic conjugate of each point being taken with respect to the two angles of the tetrahedron that are on the same edge. If the points (*i.e.* the aforesaid harmonic conjugates) in each face be joined to the opposite angles in that face, they will intersect in points denoted by (82), ('$=\frac{1}{3}$' being omitted); and, finally, if these latter points be joined to the opposite angles of the tetrahedron, the four straight lines thus drawn will intersect in the point denoted by (81), (omitting '$=\frac{1}{4}$').

Let $$\phi = \varepsilon t^2 + \zeta u^2 + \eta v^2 + \xi w^2 + 2\lambda tu + 2\mu tv + 2\nu tw$$
$$+ 2\rho uv + 2\sigma uw + 2\tau vw = 0 \ldots (84),$$

be the equation to any surface of the second degree. Multiplying (84) by ε, it may be put under the form

$$(\varepsilon t + \lambda u + \mu v + \nu w)^2 = (\lambda^2 - \varepsilon\zeta)\, u^2 \ldots + 2\,(\mu\nu - \varepsilon\tau)\, vw\,;$$

from which we see that $\varepsilon t + \lambda u + \mu v + \nu w = 0$ is the polar plane of the point $u = v = w = 0$: but

$$\frac{1}{2} \cdot \frac{d\phi}{dt} = \varepsilon t + \lambda u + \mu v + \nu w,$$

so that $\dfrac{d\phi}{dt} = 0$ is the polar plane of the point $u = v = w = 0$.

Next, to find the polar plane of any point

$$\frac{t}{\alpha} = \frac{u}{\beta} = \frac{v}{\gamma} = \frac{w}{\delta} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (85),$$

put $$\frac{U}{\beta} = \frac{u}{\beta} - \frac{t}{\alpha}, \quad \frac{V}{\gamma} = \frac{v}{\gamma} - \frac{t}{\alpha} \text{ and } \frac{W}{\delta} = \frac{w}{\delta} - \frac{t}{\alpha};$$

or $$u = U + \frac{\beta t}{\alpha}, \quad v = V + \frac{\gamma t}{\alpha}, \text{ and } w = W + \frac{\delta t}{\alpha};$$

so that the point (85) is denoted by $U = V = W = 0$: hence, if we write $U + \dfrac{\beta t}{\alpha}$, $V + \dfrac{\gamma t}{\alpha}$, and $W + \dfrac{\delta t}{\alpha}$ for u, v, and w

in (84), and then differentiate with respect to t, we shall have the polar plane required. Now the result of this, after multiplying by α, is easily seen to be

$$\alpha . \frac{d\phi}{dt} + \beta . \frac{d\phi}{du} + \gamma . \frac{d\phi}{dv} + \delta . \frac{d\phi}{dw} = 0 \text{........} (86).$$

Hence (86) is the polar plane of the point (85) with respect to the surface (84).

Now
$$\frac{1}{2} . \frac{d\phi}{dt} = \varepsilon t + \lambda u + \mu v + \nu w,$$
$$\frac{1}{2} . \frac{d\phi}{du} = \zeta u + \lambda t + \rho v + \sigma w,$$
$$\frac{1}{2} . \frac{d\phi}{dv} = \eta v + \mu t + \rho u + \tau w,$$
and
$$\frac{1}{2} . \frac{d\phi}{dw} = \xi w + \nu t + \sigma u + \tau v;$$
also put
$$\alpha' = \varepsilon\alpha + \lambda\beta + \mu\gamma + \nu\delta,$$
$$\beta' = \zeta\beta + \lambda\alpha + \rho\gamma + \sigma\delta,$$
$$\gamma' = \eta\gamma + \mu\alpha + \rho\beta + \tau\delta,$$
and
$$\delta' = \xi\delta + \nu\alpha + \sigma\beta + \tau\gamma.$$

Substitute for $\frac{d\phi}{dt}$, &c. their values just given, and (86) becomes

$$(\varepsilon\alpha + \lambda\beta + \mu\gamma + \nu\delta)\, t + (\zeta\beta + \lambda\alpha + \rho\gamma + \sigma\delta)\, u + \&c. = 0.$$

Also, from (85) we get

$$\frac{\varepsilon t + \lambda u + \mu v + \nu w}{\varepsilon\alpha + \lambda\beta + \mu\gamma + \nu\delta} = \frac{\zeta u + \lambda t + \rho v + \sigma w}{\zeta\beta + \lambda\alpha + \rho\gamma + \sigma\delta} = \ldots = \ldots$$

Simplify these equations by those that precede them, and then drop the accents. Hence the pole of the plane

$$\alpha t + \beta u + \gamma v + \delta w = 0 \text{...............} (87),^*$$

with respect to the surface (84), is

$$\frac{1}{\alpha} . \frac{d\phi}{dt} = \frac{1}{\beta} . \frac{d\phi}{du} = \frac{1}{\gamma} . \frac{d\phi}{dv} = \frac{1}{\delta} . \frac{d\phi}{dw} \text{.........} (88).$$

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* If the plane (87) touch the surface (84), (88) will denote the point of contact, and will therefore be a point in the plane. Hence, if we eliminate t, u, v, and w from (87) and (88), we shall get the condition necessary in order that (87) should be a tangent plane to the surface (84). This is the easiest way of obtaining the conditions given in the foot-note at pp. 109, 110.

These equations enable us to find the centre of the surface (84); for the centre is the pole of the plane at infinity, and, (76), the equation to the latter being

$$at + bu + cv + ew = 0,$$

we have only to write *abce* for $\alpha\beta\gamma\delta$ in (88), and we get

$$\frac{1}{a} \cdot \frac{d\phi}{dt} = \frac{1}{b} \cdot \frac{d\phi}{du} = \frac{1}{c} \cdot \frac{d\phi}{dv} = \frac{1}{e} \cdot \frac{d\phi}{dw} \quad \text{.........(89)},$$

for the equations to the centre of the surface (84).

In a similar manner it may be shewn that if

$$\psi = \varepsilon t^2 + \zeta u^2 + \eta v^2 + 2\lambda tu + 2\mu tv + 2\nu vw = 0 \text{...(90)},$$

be the equation to a cone, and

$$\frac{t}{\alpha} = \frac{u}{\beta} = \frac{v}{\gamma} \quad \text{.........................(91)},$$

those of any straight line through the vertex; then the polar plane of (91) with respect to the cone (90) is denoted by

$$\alpha . \frac{d\psi}{dt} + \beta . \frac{d\psi}{du} + \gamma . \frac{d\psi}{dv} = 0 \quad \text{........... (92)}.$$

Also, if $\qquad \alpha t + \beta u + \gamma v = 0$ .... (93),

be the equation of any plane through the vertex of (90), its polar line with respect to the cone (90) is denoted by

$$\frac{1}{\alpha} \cdot \frac{d\psi}{dt} = \frac{1}{\beta} \cdot \frac{d\psi}{du} = \frac{1}{\gamma} \cdot \frac{d\psi}{dv} \quad \text{...............(94)}.$$

Since (90), (91) and (92), also (90), (94) and (93), are cut by any plane (not passing through the vertex) in a conic, a point and a straight line respectively, such that the point is the pole of the straight line relative to the conic, it follows that if a plane cut the cone (90) in a conic, we can find the equation to the polar line of any given point in that plane relative to the conic; and conversely.

Hence also we can find the equations to the centre of the conic in which a plane, $w = 0$, cuts the cone (90). From (76) we have

$$at + bu + cv = 1 - ew, \text{ identically,}$$

so that the plane $at + bu + cv = 0$, (or $ew - 1 = 0$), which passes through the vertex of the cone, is parallel to the plane $w = 0$ of the conic, and therefore the intersection of these planes is at infinity; consequently the pole of this line with respect to the conic is its centre. Hence, writing

abc for $\alpha\beta\gamma$ in (94), we see that the equations to the centre of the conic in which the plane $w=0$ cuts the cone (90) are

$$w=0,\quad \frac{1}{a}\cdot\frac{d\psi}{dt}=\frac{1}{b}\cdot\frac{d\psi}{du}=\frac{1}{c}\cdot\frac{d\psi}{dv}\quad\text{.........(95).*}$$

Putting $w=0$ in (2), we see that the equations to the conic in which the face *ABC* is intersected by the surface (2) touching edges, are

$$w=0,\ \text{and}\ t^2+u^2+v^2-2tu-2tv-2uv=0.$$

Hence, (95), the centre of the conic is denoted by $w=0$, combined with

$$\frac{t-u-v}{a}=\frac{u-t-v}{b}=\frac{v-t-u}{c},$$

which are equivalent to

$$\frac{t}{b+c}=\frac{u}{a+c}=\frac{v}{a+b},$$

and these are evidently the equations to the straight line drawn from the angle *D* or (*tuv*) to the centre of the conic.

Proceeding in a similar manner, we find that the following are the equations to the straight lines drawn from the angular points of the tetrahedron to the centres of the conics in which the opposite faces are intersected by the surface (2).

$$\left.\begin{aligned}\frac{u}{c+e}=\frac{v}{b+e}=\frac{w}{b+c}\\ \frac{t}{c+e}=\frac{v}{a+e}=\frac{w}{a+c}\\ \frac{t}{b+e}=\frac{u}{a+e}=\frac{w}{a+b}\\ \frac{t}{b+c}=\frac{u}{a+c}=\frac{v}{a+b}\end{aligned}\right\}\quad\text{.................(96).}$$

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* This evidently amounts to saying that the plane whose equation is

$$at+bu+cv=1,$$

will cut the cone (90) in a conic whose centre is denoted by

$$\frac{1}{a}\cdot\frac{d\psi}{dt}=\frac{1}{b}\cdot\frac{d\psi}{du}=\frac{1}{c}\cdot\frac{d\psi}{dv},$$

combined of course with $at+bu+cv=1$.

Now these lines evidently lie on the hyperboloid

$$\{(b+e)(a+c)-(b+c)(a+e)\}.\{(a+b)\,tu+(c+e)\,vw\}$$
$$+\{(b+c)(a+e)-(a+b)(c+e)\}.\{(a+c)\,tv+(b+e)\,uw\}$$
$$+\{(a+b)(c+e)-(b+e)(a+c)\}.\{(a+e)\,tw+(b+c)\,uv\}=0\ldots(97);$$

and this hyperboloid contains the point $t=u=v=w$, or the point O. Hence the following theorem:

XXXIV. *Let a surface of the second degree touch the edges of a tetrahedron, and cut the faces in conics; the four straight lines drawn from the centres of the conics to the opposite angular points of the tetrahedron lie in a ruled hyperboloid and belong to the same system of generators; also the point in which intersect the three straight lines joining the points of contact of opposite edges, is a point in this hyperboloid.*

By using (94) instead of (95) it is easily seen that this theorem is still true when instead of the centres of the conics we substitute the poles of the lines in which the faces are intersected by *any* plane, the pole of each line being taken with respect to the conic in the same face of the tetrahedron. It is worthy of observation too that, though the hyperboloid will vary for different positions of the cutting plane, yet it (the hyperboloid) always passes through the fixed point in which intersect the three straight lines joining the points of contact of opposite edges.

Again, the polar plane of the point

$$\frac{t}{\alpha}=\frac{u}{\beta}=\frac{v}{\gamma}=\frac{w}{\delta}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (98),$$

that is, of the straight line

$$\frac{t}{\alpha}=\frac{u}{\beta}=\frac{v}{\gamma},$$

with respect to the circumscribed cone (of the given tetrahedron)

$$tu+tv+uv=0$$

is, (92), $\quad \alpha(u+v)+\beta(t+v)+\gamma(t+u)=0;$

that is, $\quad (\beta+\gamma)\,t+(\alpha+\gamma)\,u+(\alpha+\beta)\,v=0.$

In a similar manner the polar planes of the point (98) with respect to the other circumscribed cones may be obtained, and tabulating the whole, we have

$$\left.\begin{array}{l}(\gamma+\delta)\,u+(\beta+\delta)\,v+(\beta+\gamma)\,w=0\\(\gamma+\delta)\,t+(\alpha+\delta)\,v+(\alpha+\gamma)\,w=0\\(\beta+\delta)\,t+(\alpha+\delta)\,u+(\alpha+\beta)\,w=0\\(\beta+\gamma)\,t+(\alpha+\gamma)\,u+(\alpha+\beta)\,v=0\end{array}\right\}\ldots\ldots(99).$$

The four straight lines in which these planes intersect the faces t, u, v, w respectively of the tetrahedron evidently lie in the hyperboloid whose equation is

$$(\gamma+\delta)(\beta+\delta)(\beta+\gamma)t^2+(\gamma+\delta)(\alpha+\delta)(\alpha+\gamma)u^2$$
$$+(\beta+\delta)(\alpha+\delta)(\alpha+\beta)v^2+(\beta+\gamma)(\alpha+\gamma)(\alpha+\beta)w^2$$
$$+\{(\beta+\delta)(\alpha+\gamma)+(\beta+\gamma)(\alpha+\delta)\}.\{(\gamma+\delta)tu+(\alpha+\beta)vw\}$$
$$+\{(\gamma+\delta)(\alpha+\beta)+(\beta+\gamma)(\alpha+\delta)\}.\{(\beta+\delta)tv+(\alpha+\gamma)uw\}$$
$$+\{(\gamma+\delta)(\alpha+\beta)+(\beta+\delta)(\alpha+\gamma)\}.\{(\beta+\gamma)tw+(\alpha+\delta)uv\}=0\ldots(100).$$

If we put

$$X=(-\alpha+\beta+\gamma+\delta)t+(\gamma+\delta)u+(\beta+\delta)v+(\beta+\gamma)w$$

and

$$Y=(\alpha^2+\beta\gamma+\beta\delta+\gamma\delta)t+(\alpha+\gamma)(\alpha+\delta)u+(\alpha+\beta)(\alpha+\delta)v+(\alpha+\beta)(\alpha+\gamma)w,$$

it will be found that (100) may be made to assume the form

$$(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(t+u+v+w)t+XY=0,$$

and hence the hyperboloid (100) always touches the plane $t+u+v+w=0$, whatever be the values of $\alpha\beta\gamma\delta$, that is, whatever be the position of the point (98). Hence the following theorem:

XXXV. *Let a surface of the second degree touch the edges of a tetrahedron, and the four circumscribed cones be drawn; also let the polar plane of any point be taken with respect to each of these cones: these four planes will intersect the corresponding faces in four straight lines belonging to the same system of generators in a ruled hyperboloid: and this hyperboloid, whatever be the position of the point whose polar planes are taken, always touches the fixed plane in which intersect the tangent planes to the first-mentioned surface that pass through opposite edges of the tetrahedron.*

This theorem includes some interesting particular cases, one of which I may mention. We have seen that (100) may be thrown into any of the forms

$$(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(t+u+v+w)t+XY=0,$$
$$(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(t+u+v+w)u+X'Y'=0,$$
$$(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(t+u+v+w)v+X''Y''=0,$$
$$(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(t+u+v+w)w+X'''Y'''=0;$$

so that when any two of the quantities $\alpha, \beta, \gamma, \delta$ are equal, the hyperboloid will split up into two planes; and this will be the case when the point whose polar planes are taken is situated in any of the planes (9...14).

By (89) the equations to the centre of the surface (2) are

$$\frac{t-u-v-w}{a}=\frac{u-t-v-w}{b}=\frac{v-t-u-w}{c}=\frac{w-t-u-v}{e},$$

which, (77), are equivalent to either system of equations

$$\frac{t}{a'}=\frac{u}{b'}=\frac{v}{c'}=\frac{w}{e'} \text{ ....(101)},$$

or

$$\frac{t'}{a}=\frac{u'}{b}=\frac{v'}{c}=\frac{w'}{e} \text{ .... (102)}.$$

The centre of the surface (2) or (38) is also, (80, 89), denoted by

$$-\frac{p}{a''}=\frac{q}{b''}=\frac{r}{c''}=\frac{s}{e''} \text{ ....(103)}.$$

By (96) the equations to the centre of the conic in which the face t' cuts the surface (2) are

$$t'=0, \text{ and } \frac{u'}{c'+e'}=\frac{v'}{b'+e'}=\frac{w'}{b'+c'},$$

which are equivalent to

$$t'=0, \text{ and } \frac{t'-u'+v'+w'}{b'}=\frac{t'+u'-v'+w'}{c'}=\frac{t'+u'+v'-w'}{e'};$$

that is, $$t'=0, \text{ and } \frac{u}{b'}=\frac{v}{c'}=\frac{w}{e'}.$$

Now these equations, the first of (96), equations (101), the equations $u=v=w=0$, and the first of (39), each satisfy the equation

$$(c-e)\,u+(e-b)\,v+(b-c)\,w=0,$$

or, (which is the same thing, (77),)

$$(e'-c')\,u+(b'-e')\,v+(c'-b')\,w=0,$$

and hence we have the following theorem (which is due to the late Mr. Hearn),

XXXVI. *The centres of the two conics in which the surface* (2) *cuts any two corresponding faces* (*as BCD, B'C'D'*) *of the two tetrahedra, the centre of the surface itself, and the two corresponding angular points* (*A, A'*), *are in one plane.*

By (89) the centres of the surfaces (61) and (62) are denoted by

$$-\frac{p}{3a''}=\frac{q}{b''}=\frac{r}{c''}=\frac{s}{e''},$$

and $$-\frac{3p}{a''}=\frac{q}{b''}=\frac{r}{c''}=\frac{s}{e''}, \text{ respectively.}$$

Now each of these systems of equations, as well as (103), and $q = r = s = 0$ (which denote the point O), satisfies the equations

$$\frac{q}{b''} = \frac{r}{c''} = \frac{s}{e''}.$$

Hence

XXXVII. *The point of intersection of straight lines joining the corresponding angles of the two tetrahedra, and the centres of the surface* (2) *touching the edges, of the surface* (61) *inscribed in the inscribed cones, and of the surface* (62) *circumscribed about the circumscribed cones, are in a straight line.*

The equations to the straight line joining the middle points of the edges AB and CD are, (83),

$$at = bu \text{ and } cv = ew:$$

also the equations to that joining the middle points of $A'B'$ and $C'D'$ are

$$a't' = b'u' \text{ and } c'v' = e'w';$$

that is,

$$(-a+b+c+e)(-t+u+v+w) = (a-b+c+e)(t-u+v+w),$$

and $(a+b-c+e)(t+u-v+w) = (a+b+c-e)(t+u+v-w)$.

Now the equations to both these straight lines are satisfied by the equations

$$\frac{t}{b} = \frac{u}{a} = \frac{v}{e} = \frac{w}{c},$$

which denote the centre of the first hyperboloid (57).

Hence, recollecting that the centres of gravity of the two tetrahedra are respectively in the beforementioned straight lines, we have the following theorem:

XXXVIII. *The straight line joining the middle points of any two opposite edges (as AB, CD) of the tetrahedron $ABCD$, and that joining the middle points of the corresponding pair ($A'B'$, $C'D'$) of the tetrahedron $A'B'C'D'$, intersect in the centre of the hyperboloid passing through the other eight edges.* Hence also *the middle points of any pair of opposite edges of the given tetrahedron, those of the corresponding pair of the other tetrahedron, the centres of gravity of the two tetrahedra, and the centre of the hyperboloid passing through the other eight edges, are in one plane.*

A great number of theorems somewhat similar to the last five might be given, but I shall content myself with only one more.

The centres of the surfaces (64), (65), (66) and (67) are denoted as follows:

$$-\frac{p}{a''} = \frac{q}{b''} = \frac{r}{c''} = \frac{s}{e''},$$

$$\frac{p}{a''} = -\frac{q}{b''} = \frac{r}{c''} = \frac{s}{e''},$$

$$\frac{p}{a''} = \frac{q}{b''} = -\frac{r}{c''} = \frac{s}{e''},$$

and $$\frac{p}{a''} = \frac{q}{b''} = \frac{r}{c''} = -\frac{s}{e''} \text{ respectively;}$$

hence the straight lines drawn from these points to the corresponding angles of the tetrahedron $OO'O''O'''$ are

$$\frac{q}{b''} = \frac{r}{c''} = \frac{s}{e''},$$

$$\frac{p}{a''} = \frac{r}{c''} = \frac{s}{e''},$$

$$\frac{p}{a''} = \frac{q}{b''} = \frac{s}{e''},$$

and $$\frac{p}{a''} = \frac{q}{b''} = \frac{r}{c''} \text{ respectively;}$$

and these intersect in the point

$$\frac{p}{a''} = \frac{q}{b''} = \frac{r}{c''} = \frac{s}{e''}.$$

Hence

XXXIX. *The straight lines joining the centres of the surfaces* (64), (65), (66) *and* (67) (*which touch the edges of the two tetrahedra* $ABCD$ *and* $A'B'C'D'$ *at the points of intersection of these edges*), *to the corresponding angles of the third tetrahedron* $OO'O''O'''$ *intersect in a point.*

It is scarcely necessary to observe that these theorems admit of being generalized (by substituting *poles* for *centres*, &c.)

I come, finally, to the subject alluded to at page 118.

In my third memoir 'On the Theorems in Space analogous to those of Pascal and Brianchon in a Plane,' (*Journal*, new series, vol. VI. p. 132), I have shewn under what conditions, at once necessary and sufficient, a duodecangular octahedron

will be inscriptible in, and an octangular dodecahedron* circumscriptible about, a surface of the second degree, namely, in the former case that the four straight lines in which the opposite faces intersect, and in the latter that the four straight lines joining opposite angles, shall lie in a ruled hyperboloid and belong to the same system of generators. We cannot, of course, inquire under what conditions a duodecangular octahedron will be circumscriptible about, or an octangular dodecahedron inscriptible in, a surface of the second degree, because innumerable surfaces may evidently be always inscribed in the former, and circumscribed about the latter:† but we may inquire under what conditions, at once necessary and sufficient, a surface of the second degree may touch the edges of these figures; and these conditions it is my present object to ascertain.

I begin by finding the equations to the faces of a duodecangular octahedron whose edges touch a surface of the second degree.

Let $t=0$, $u=0$, $v=0$, $w=0$, be the equations to the hexagonal faces of a duodecangular octahedron; then, supposing t, u, v, and w to have been multiplied by the proper constants, the equation to the surface of the second degree touching the edges of the duodecangular octahedron may be denoted by

$$t^2+u^2+v^2+w^2-2tu-2tv-2tw-2uv-2uw-2vw=0\ldots(a),$$

for the edges of the tetrahedron $(tuvw)$ are edges of the octahedron.

Let $$T=t+\beta u+\gamma v+\delta w=0$$

* To avoid unnecessary reference, I copy the definitions of these two solid figures from the page cited in the text.

1. A *duodecangular octahedron* is a solid figure generated by taking a tetrahedron, and cutting off a portion towards each angular point by a plane.

2. An *octangular dodecahedron* is a solid figure generated by taking four tetrahedra whose *bases* are respectively equal to the faces of a fifth tetrahedron, and applying the bases of the former to the faces equal to them of the latter.

It is obvious that the octahedron has four hexagonal and four triangular faces, and the dodecahedron four hexahedral and four trihedral angles.

† We may, however, ask under what conditions the faces of duodecangular octahedron are the eight common tangent planes, and the angular points of an octangular dodecahedron the eight points of intersection, of three surfaces of the second degree. It is easy by the first four theorems of my second memoir 'On the Theorems in Space analogous to those of Pascal and Brianchon in a Plane,' (*Journal*, new series, vol. v. pp. 58 and 60) to give these conditions, which, however, may be said to belong rather to the system of planes or points than to the solid figure.

denote the triangular face of the octahedron opposite to the hexagonal face t. Since the edge (uT) or $u = t + \gamma v + \delta w = 0$ touches the surface (a), putting $u = 0$ and $t = -\gamma v - \delta w$ in (a), the resulting equation

$$(\gamma + 1)^2 v^2 + (\delta + 1)^2 w^2 + 2\{(\gamma + 1)(\delta + 1) - 2\} vw = 0$$

must be a complete square. This requires that

$$(\gamma + 1)^2 (\delta + 1)^2 = \{(\gamma + 1)(\delta + 1) - 2\}^2,$$

or

$$(\gamma + 1)(\delta + 1) = 1.$$

Proceeding in a similar manner, we find that

$$(\beta + 1)(\delta + 1) = 1,$$

and

$$(\beta + 1)(\gamma + 1) = 1,$$

if the edges (vT) and (wT) touch the surface (a).

These three equations being solved give

either

$$\beta = \gamma = \delta = 0,$$

or

$$\beta = \gamma = \delta = -2,$$

the former of which being inadmissible, we must take the latter, and then we have

$$-T = -t + 2u + 2v + 2w,$$

so that

$$-\tfrac{1}{2}t + u + v + w = 0$$

is the equation to the triangular face opposite to the hexagonal face t. In a similar manner the equations to the other triangular faces may be found, and collecting the whole, we have

$$\left.\begin{aligned} -\tfrac{1}{2}t + u + v + w &= 0 \\ -\tfrac{1}{2}u + t + v + w &= 0 \\ -\tfrac{1}{2}v + t + u + w &= 0 \\ -\tfrac{1}{2}w + t + u + v &= 0 \end{aligned}\right\} \ldots\ldots\ldots (104).$$

Hence, if the edges of a duodecangular octahedron touch a surface of the second degree, its hexagonal faces may be denoted by $t = 0$, $u = 0$, $v = 0$ and $w = 0$, its triangular faces by (104), and the surface itself by (2) or (a).

Now it is evident at a glance that the opposite faces intersect in straight lines in the plane $t + u + v + w = 0$, but this implies only *five* conditions; whereas, that a surface of the second degree should touch the edges (eighteen in number) of a duodecangular octahedron requires *nine* conditions. The four conditions wanting can, however, be supplied as follows. It is evident that each hexagonal force is circumscribed about

a conic, so that the diagonals joining the opposite angles of each will intersect in a point (by Brianchon's theorem); and since there are four hexagonal faces, this will supply four conditions. We have, therefore, the following theorem:

XL. *If a surface of the second degree touch the edges of a duodecangular octahedron, the opposite faces will intersect in four straight lines in one plane, and the three diagonals joining the opposite angles of each hexagonal face will intersect in a point.*

It remains to be seen whether the conditions found are sufficient as well as necessary; or, in other words, whether the preceding theorem is convertible. I shall first enunciate the converse theorem, and then shew that it is true.

XLI. *If the opposite faces of a duodecangular octahedron intersect in four straight lines in one plane; and if, moreover, the three diagonals joining the opposite angles of each hexagonal face intersect in a point; the edges of the octahedron will touch a surface of the second degree.*

Let $t = 0$, $u = 0$, $v = 0$, and $w = 0$, be the equations to the hexagonal faces of the octahedron; then, supposing t, u, v, and w to have been multiplied by the proper constants, the equation to the plane in which the opposite faces intersect may be denoted by

$$t + u + v + w = 0;$$

hence the triangular faces will be denoted as follows:

$$\left.\begin{array}{l} T = \alpha t + u + v + w = 0 \\ U = t + \beta u + v + w = 0 \\ V = t + u + \gamma v + w = 0 \\ W = t + u + v + \delta w = 0 \end{array}\right\} \ldots\ldots\ldots (b),$$

where it is evident that neither α, β, γ nor δ can $= 1$.

The equations to the angular points in the hexagonal face t are $t = 0$, combined with $u = V = 0$, $u = W = 0$, $v = W = 0$, $v = U = 0$, $w = U = 0$, and $w = V = 0$ respectively; and it is hence easy to see that the equations to the three diagonals joining opposite angles are $t = 0$, combined with

$$u + \gamma v + \delta w = 0,$$

$$\beta u + v + \delta w = 0,$$

and $$\beta u + \gamma v + w = 0,$$

respectively. Now these diagonals pass through the same point; hence, eliminating v and w from the last three equations, we must have

$$1 - \gamma\delta - \beta\gamma - \beta\delta + 2\beta\gamma\delta = 0 \ldots\ldots\ldots (c).$$

Proceeding in a similar manner with the other hexagonal faces, we obtain

$$1 - \gamma\delta - \alpha\gamma - \alpha\delta + 2\alpha\gamma\delta = 0 \ldots\ldots\ldots (e),$$

$$1 - \beta\delta - \alpha\beta - \alpha\delta + 2\alpha\beta\delta = 0,$$

and $$1 - \beta\gamma - \alpha\beta - \alpha\gamma + 2\alpha\beta\gamma = 0.$$

Deduct (*e*) from (*c*); therefore

$$(\alpha - \beta)(\gamma + \delta - 2\gamma\delta) = 0,$$

hence, either $\alpha = \beta$ or $\gamma + \delta - 2\gamma\delta = 0$.

Taking the latter, multiply it by β and add the product to (*c*),

therefore $$1 - \gamma\delta = 0;$$

but the two equations $\gamma + \delta - 2\gamma\delta = 0$ and $1 - \gamma\delta = 0$ give $\gamma = \delta = 1$, which are inadmissible values; hence we must have $\alpha = \beta$. Treating every two of the four equations in the same way, we find

$$\alpha = \beta = \gamma = \delta,$$

and (*c*) is reduced to

$$1 - 3\alpha^2 + 2\alpha^3 = 0.$$

The three roots of this equation are 1, 1, and $-\frac{1}{2}$, the last only being an available root, so that

$$\alpha = \beta = \gamma = \delta = -\tfrac{1}{2}.$$

Hence the equations (*b*) coincide with the equations (104), and all the edges of the duodecangular octahedron therefore touch the surface (2) or (*a*).

It remains to consider this subject for the octangular dodecahedron. Here it would be shortest to employ the method of reciprocal polars, but I shall give an independent investigation.

Let $t = 0$, $u = 0$, $v = 0$, $w = 0$, be the equations to those diagonal planes of the octangular dodecahedron that pass through the four hexahedral angles;* then, if the edges of the dodecahedron touch a surface of the second degree, the equation to the surface may be denoted by (*a*), for the edges of the tetrahedron (*tuvw*) are edges of the dodecahedron. Also, since the faces intersecting in that trihedral angle which is opposite to the hexahedral angle (*uvw*) pass through the

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* In other words, $t=0$, $u=0$, $v=0$, $w=0$, denote the faces of the 'fifth tetrahedron' mentioned in the definition at the foot of p. 142.

edges (tu), (tv), and (tw) respectively, their equations may be denoted by

$$u = \alpha t, \quad v = \beta t, \quad \text{and } w = \gamma t.$$

Since the edge in which the faces $u = \alpha t$ and $v = \beta t$ intersect touches the surface (a), if we write αt and βt in (a) for u and v respectively, the resulting equation

$$(1 + \alpha^2 + \beta^2 - 2\alpha - 2\beta - 2\alpha\beta)\, t^2 + w^2 - 2(1 + \alpha + \beta)\, tw = 0,$$

must be a complete square. This requires

$$1 + \alpha^2 + \beta^2 - 2\alpha - 2\beta - 2\alpha\beta = (1 + \alpha + \beta)^2,$$

which reduces to

$$(\alpha + 1)(\beta + 1) = 1.$$

In a similar manner, we get

$$(\alpha + 1)(\gamma + 1) = 1,$$

and
$$(\beta + 1)(\gamma + 1) = 1.$$

These equations being solved, we have

either
$$\alpha = \beta = \gamma = 0,$$

or
$$\alpha = \beta = \gamma = -2.$$

The former values being clearly inadmissible, we must take the latter and the equations to the three faces become $u + 2t = 0$, $v + 2t = 0$, and $w + 2t = 0$. In a similar manner the equations to the other faces may be obtained, and collecting the whole we see that if the edges of an octangular dodecahedron touch a surface of the second degree, its faces may be denoted as follows:

$$\left.\begin{array}{lll} u + 2t = 0, & v + 2t = 0, & w + 2t = 0 \\ t + 2u = 0, & v + 2u = 0, & w + 2u = 0 \\ t + 2v = 0, & u + 2v = 0, & w + 2v = 0 \\ t + 2w = 0, & u + 2w = 0, & v + 2w = 0 \end{array}\right\} \ldots(105);$$

where the equations to those faces that intersect in a trihedral angle are placed in the same horizontal line. Hence, eliminating t, u, v, and w respectively from the first, second, third, and fourth rows of equations, we get the equations to the straight lines joining the opposite angles of the dodecahedron, namely,

$$u = v = w,$$
$$t = v = w,$$
$$t = u = w,$$

and
$$t = u = v,$$

and these pass through the point $t = u = v = w$.

Also the equations to the faces that intersect in one of the hexahedral angles, as (uvw), are

$$u + 2v = 0, \quad u + 2w = 0,$$
$$v + 2w = 0, \quad v + 2u = 0,$$

and
$$w + 2u = 0, \quad w + 2v = 0,$$

where the equations to the opposite faces are placed in the same horizontal line. Hence, the opposite faces intersect in straight lines in the plane $u + v + w = 0$; so that the following theorem is established.

XLII. *If a surface of the second degree touch the edges of an octangular dodecahedron, the four straight lines joining the opposite angles will intersect in one point, and the three straight lines in which the opposite faces of each hexahedral angle intersect, will be in one plane.*

Conversely,

XLIII. *If the straight lines joining the opposite angles of an octangular dodecahedron intersect in a point, and if besides the three straight lines in which the opposite faces of each hexahedral angle intersect, lie in a plane; then shall the edges of the octangular dodecahedron touch a surface of the second degree.*

To prove this theorem, let as before $t = 0$, $u = 0$, $v = 0$, $w = 0$, denote the diagonal planes passing through every three of the hexahedral angular points: also let

$$t = u = v = w$$

be the equations to the point in which intersect the straight lines joining the opposite angles. One of these lines (namely that passing through the hexahedral angle (uvw)) is denoted by

$$u = v = w;$$

so that if a be properly assumed,

$$\frac{t}{a} = u = v = w$$

will denote the trihedral angular point on this line, and the equations to the three faces intersecting in this point will be $t = au$, $t = av$, and $t = aw$. In like manner the equations to the other faces will be obtained, and collecting the whole we have

$$\left.\begin{array}{lll} t = au, & t = av, & t = aw \\ u = \beta t, & u = \beta v, & u = \beta w \\ v = \gamma t, & v = \gamma u, & v = \gamma w \\ w = \delta t, & w = \delta u, & w = \delta v \end{array}\right\} \ldots\ldots\ldots (f).$$

Since none of the angles of the dodecahedron must coincide with the point $t = u = v = w$, it is clear that none of the quantities α, β, γ, or δ can $= 1$.

Now the faces that intersect in the hexahedral angle (uvw) are

$$v = \gamma u, \quad w = \delta u,$$

$$w = \delta v, \quad u = \beta v,$$

and

$$u = \beta w, \quad v = \gamma w,$$

where the equations to the opposite faces are placed in the same horizontal line. Hence, these opposite faces intersect in the straight lines

$$u = \frac{v}{\gamma} = \frac{w}{\delta},$$

$$v = \frac{u}{\beta} = \frac{w}{\delta},$$

and

$$w = \frac{u}{\beta} = \frac{v}{\gamma};$$

and if these straight lines are in the plane $lu + mv + nw = 0$, we must have

$$l + \gamma m + \delta n = 0,$$

$$\beta l + m + \delta n = 0,$$

and

$$\beta l + \gamma m + n = 0;$$

and eliminating l and m from these equations, we get

$$1 - \gamma\delta - \beta\gamma - \beta\delta + 2\beta\gamma\delta = 0.$$

Similarly, we get $1 - \gamma\delta - \alpha\gamma - \alpha\delta + 2\alpha\gamma\delta = 0,$

$$1 - \beta\delta - \alpha\beta - \alpha\delta + 2\alpha\beta\delta = 0,$$

and $1 - \beta\gamma - \alpha\beta - \alpha\gamma + 2\alpha\beta\gamma = 0.$

Now these equations have been already solved, and since in this case also none of the quantities α, β, γ, δ can $= 1$, we must have

$$\alpha = \beta = \gamma = \delta = -\tfrac{1}{2}.$$

These values make (f) coincide with (105), and hence the edges of the octangular dodecahedron will touch the surface (a) of the second degree.

York Town, near Bagshot,
November 30th, 1852.

ON A PROBLEM IN COMBINATIONS.

By R. R. Anstice.

(*Continued from* Vol. VII. p. 292).

To complete the discussion of the problem in combinations which I have begun, I will now give the general expression, a particular case of which was considered in my former paper, and add a sketch of the proof.

Let n be an odd number; let $3.2^{s+1}.n+1$ be a prime; and let r be a primitive root thereto. Let ρ, ρ_1 be any two different roots of the equivalence

$$x^{3n}+1 \equiv 0 \quad (\text{modulus } 3.2^{s+1}.n+1).$$

(This same modulus will be understood in all the equivalences in this paper where no other is expressed.)

Let α be any odd number less than 2^{s+1}. Also, as before, let k be the constant term, and P_0P_1, &c., Q_0Q_1, &c. the successive members of the two cycles. Then primary arrangement

$$= kP_0Q_0$$

$$+ \Sigma\Sigma P_{\frac{\rho_1-\rho}{\rho_1-1}r^{2^{s+1}i+\alpha i'}},\quad Q_{r^{2^{s+1}i+\alpha i'}},\quad Q_{\rho r^{2^{s+1}i+\alpha i'}}$$

from $i=0$ to $i=3n-1$, and from $i'=0$ to $i'=2^s-1$,

$$+ \Sigma\Sigma P_{\frac{\rho_1-\rho}{\rho_1-1}r^{2^s(2i+1)+\alpha i'}},\quad P_{\frac{\rho_1-\rho}{\rho_1-1}r^{2^s(2i+2n+1)+\alpha i'}},\quad P_{\frac{\rho_1-\rho}{\rho_1-1}r^{2^s(2i+4n+1)+\alpha i'}}$$

from $i=0$ to $i=n-1$, and from $i'=0$ to $i'=2^s-1$.

First, to the letter P all the subscripts are annexed, *i.e.* none are repeated. These subscripts are the different terms of the series

$$\Sigma\Sigma\frac{\rho_1-\rho}{\rho_1-\rho}r^{2^sj+\alpha i'}$$

from $j=0$ to $j=6n-1$, and from $i'=0$ to $i'=2^s-1$.

If possible then, let

$$\frac{\rho_1-\rho}{\rho_1-1}r^{2^sj+\alpha i'} \equiv \frac{\rho_1-\rho}{\rho_1-1}r^{2^sJ+\alpha I'},$$

where J and I' are comprised between the same limits as j and i'.

Therefore $\qquad r^{2^s(J-j)+\alpha(I'-i')} \equiv 1,$

and, consequently,

$$2^s(J-j)+\alpha(I'-i') \equiv 3.2^{s+1}.n\lambda.$$

λ being some integer. Therefore $I' - i'$ must be divisible by 2^s. But the difference $I' - i'$ lies between the limits (inclusive) $\pm(2^s - 1)$, and therefore cannot be divisible by 2^s unless it vanishes.

Let then $I' - i' = 0$, and divide both sides by 2^s. Therefore $J - j = 6n\lambda$. Here the value $\lambda = 0$ is inadmissible, since we cannot have $J = j$ and $I' = i'$ together. And so is any other value; for the difference $J - j$ lies between the limits (inclusive) $\pm(6n - 1)$. So the above equation is impossible.

Next, to the letter Q all the subscripts are annexed: *i.e.* none are repeated. The possibility of the repetition of any of the subscripts of Q is easily seen to depend on the possibility of any of the three equivalences

$$\left.\begin{aligned} r^{2^{s+1}i+\alpha i'} &\equiv r^{2^{s+1}I+\alpha I'} \\ \rho . r^{2^{s+1}i+\alpha i'} &\equiv \rho . r^{2^{s+1}I+\alpha I'} \\ \rho . r^{2^{s+1}i+\alpha i'} &\equiv r^{2^{s+1}I+\alpha I'} \end{aligned}\right\}.$$

It being given that both i and I are less than $3n$; that both i' and I' are less than 2^s; and further, that we cannot have simultaneously $i = I$ and $i' = I'$.

The two first of these equivalences are manifestly impossible from the same reasoning as before. From the third we get
$$\rho \equiv r^{2^{s+1}(I-i)+\alpha(I'-i')}.$$

But, by the definition of ρ, its most general value, expressed as a power of r, must be
$$\rho \equiv r^{2^s(2\lambda+1)},$$

λ being some integer. Therefore
$$2^s(2\lambda + 1) \equiv 2^{s+1}(I - i) + \alpha(I' - i').$$

Impossible (as before) unless $I' - i' = 0$. And if this is the case, still impossible, since then we must have
$$2\lambda + 1 = 2(I - i),$$

or an odd equal to an even number.

Again, the differences of the subscripts of duads of the second class cannot be equivalent, whether estimated positively or negatively. The possibility of their being equivalent is easily seen to rest on the possibility of the equivalences
$$(\rho - 1)\, r^{2^{s+1}I+\alpha I'} \equiv \pm(\rho - 1)\, r^{2^{s+1}i+\alpha i'};$$

i.e. of the equivalences
$$r^{2^{s+1}(I-i)+\alpha(I'-i')} \equiv \pm 1,$$

with the same conditions as before.

If we take the upper sign, we must have

$$2^{r+1}(I-i)+\alpha(I'-i')\equiv 3.2^{r+1}.n\lambda,$$

λ being some integer. Impossible (as before) unless $I'-i'=0$. Let then $I'-i'=0$ and divide both sides by 2^{r+1}.

Therefore $I-i=3n\lambda$, which is impossible, since the difference $I-i$ lies between the limits (inclusive) $\pm(3n-1)$, and the value $\lambda=0$ is inadmissible, as before.

If we take the lower sign, we must have

$$2^{r+1}(I-i)+\alpha(I'-i')=3.2^r n+3.2^{r+1}.n\lambda,$$

λ being some integer. Impossible (as before) unless $I'-i'=0$. And if this is the case, still impossible, since then we must have

$$2(I-i)=3n+6n\lambda,$$

i.e. an even number equal to an odd one.

The rest of the proof may be easily inferred.

The number of distinct species remains to be considered.

Now in the general expression, as it stands, we give to the variables i and i' successive integral values, beginning from 0 in each case. But in proving the efficiency of the cycles, we are only concerned with the *differences* $I-i$ and $I'-i'$. Therefore we might have taken for i or for i' successive integers, commencing from *any* integer as origin, and the cycles would still have been efficient. Now what effect does this change of origin of i or of i' have on the form of the general expression?

First, suppose the origin of i is changed. This will have no effect whatever. It is true that i is summed in one case between limits 0 and $3n-1$, and in another between limits 0 and $n-1$. But we might have used in this case the same limits as in the former and divided the result by 3, as we should only have reproduced the same triads. Now if we attribute to i any value equal to or greater than $3n$, only its *remainder* when divided by $3n$ will have any effect, since i is always multiplied by 2^{r+1} and $r^{3.2^{r+1}n}\equiv 1$.

Next, suppose the origin of i' is changed. Suppose that successive integral values are to be substituted for i' beginning from λ instead of from 0. This will clearly be the same as if we wrote $i'+\lambda$ in place of i' in the general expression, and then measured the i''s from origin 0 as before. This certainly may change the form of the expression, but not the system generated, since it is the same change as would be wrought by multiplying all the subscripts by the

same factor $r^{\alpha\lambda}$. Consequently we do not extend the expression by altering the origin of i'.

Again, the efficiency of the cycles would have been demonstrated in just the same way, if we had taken for α any odd number whatever. What effect then is produced on the form of the general expression by taking for α odd numbers greater than 2^{s+1}?

In place of α write $2^{s+1}\lambda + \alpha$; and the same effect is produced as by writing in place of i, $i + \lambda i'$; that is, it has the effect only of changing the origin of i, and therefore produces no alteration in the general expression.

Or again, if we take another primitive root; if we write r^β in place of r, where β is any number prime to $6n$, it is easily seen (though the explanation would be rather tedious), that though we may change the form of the primary arrangement, it would be only for another included in the same expression.

Lastly, what effect is produced on the form of the primary arrangement by multiplying all the subscripts by any the same number prime to the modulus, an operation which will not alter the system generated? Such a number must always be equivalent to some power of r. Let it $\equiv r^\mu$.

(1). Let μ be divisible by 2^{s+1}. Then the origin of i only will be altered and no effect result.

(2). Let μ be divisible by 2^s but not by 2^{s+1}. Then the same effect results as there would from the simultaneous change of ρ into $\frac{1}{\rho}$ and of ρ_1 into $\frac{1}{\rho_1}$. As will readily appear if we write $2^s(2\mu_1 + 1)$ in place of μ, and put for ρ its value $r^{2^s(2\lambda+1)}$.

(3). Let μ be not divisible by 2^s. Then the form of the primary arrangement is altered entirely. The effect produced is a change in the origin of i', by a quantity which $\equiv \frac{\mu}{\alpha}$ (modulus 2^{s+1}).

It follows from these considerations that the number of distinct species = number of odd integers less than 2^{s+1} multiplied by half the number of permutations of $3n$ things, taken in pairs $= 2^{s-1}.3n(3n-1)$.

If $s = 0$ this becomes $\frac{3n(3n-1)}{2}$, as was given in my former paper.

As an example, let us apply the formula to 27 symbols. Here $n = 1$, $s = 1$, $3.2^{s+1}n + 1 = 13$ and is prime. A primitive root of 13 is 6. Taking then $r = 6$ throughout, we can construct six forms of the primary arrangement which will generate six distinct systems, thus:

I. $(\rho \equiv r^3 \equiv 10,\ \rho_1 = -1,\ \alpha = 1)$

$$\begin{aligned} kp_0q_0 &+ p_{12}q_1q_{10} + p_4q_9q_{12} + p_{10}q_3q_4 \\ &+ p_7q_6q_8 \ + p_{11}q_2q_7 + p_8q_5q_{11} \\ &+ p_3p_1p_9 \ + p_5p_6p_2. \end{aligned}$$

II. $(\rho = 10,\ \rho_1 = -1,\ \alpha = 3)$

$$\begin{aligned} kp_0q_0 &+ p_{12}q_1q_{10} + p_4q_9q_{12} + p_{10}q_3q_4 \\ &+ p_5q_8q_2 \ + p_6q_7q_5 \ + p_2q_{11}q_6 \\ &+ p_3p_1p_9 \ + p_{11}p_8p_7. \end{aligned}$$

III. $(\rho = 10,\ \rho_1 \equiv \dfrac{1}{\rho} \equiv 4,\ \alpha = 1)$

$$\begin{aligned} kp_0q_0 &+ p_{11}q_1q_{10} + p_8q_9q_{12} \ + p_7q_3q_4 \\ &+ p_1q_6q_8 \ + p_9q_2q_7 \ + p_3q_5q_{11} \\ &+ p_6p_2p_5 \ + p_{10}p_{12}p_4. \end{aligned}$$

IV. $(\rho = 10,\ \rho_1 = 4,\ \alpha = 3)$

$$\begin{aligned} kp_0q_0 &+ p_{11}q_1q_{10} + p_8q_9q_{12} + p_7q_3q_4 \\ &+ p_{10}q_8q_2 \ + p_{12}q_7q_5 + p_4q_{11}q_6 \\ &+ p_6p_2p_5 \ + p_9p_3p_1. \end{aligned}$$

V. $(\rho = -1,\ \rho_1 = 10,\ \alpha = 1)$

$$\begin{aligned} kp_0q_0 &+ p_7q_1q_{12} + p_{11}q_9q_4 \ + p_8q_3q_{10} \\ &+ p_3q_6q_7 \ + p_1q_2q_{11} \ + p_9q_5q_8 \\ &+ p_5p_6p_2 + p_4p_{10}p_{12}. \end{aligned}$$

VI. $(\rho = -1,\ \rho_1 = 10,\ \alpha = 3)$

$$\begin{aligned} kp_0q_0 &+ p_7q_1q_{12} + p_{11}q_9q_4 + p_8q_3q_{10} \\ &+ p_4q_8q_5 \ + p_{10}q_7q_6 + p_{12}q_{11}q_2 \\ &+ p_5p_6p_2 + p_1p_9p_3. \end{aligned}$$

We might, indeed, give other forms of the primary arrangement than these, but they would all generate one or other of these systems. These are all, I mean, which can result from the formula given: for the method of Mr. Mease

and Mr. Kirkman, I should imagine, would generate distinct systems from these.

November 20, 1852.

NOTE.—I have determined, I believe, in this and my former paper, the number of distinct species comprised in the formulæ given. But I now find there are more general formulæ, including the formulæ given as particular cases: and three formulæ in particular, each comprising many distinct species, which are applicable even when the modulus is composite, provided all its factors are of the form $6n+1$.

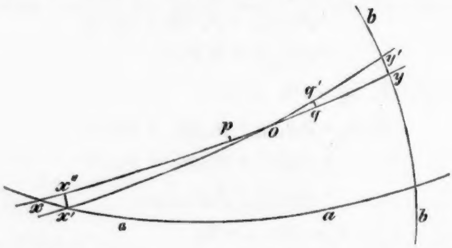
April 13, 1853.

ELEMENTARY INVESTIGATION OF THE FORMULÆ FOR THE VARIATIONS OF THE INCLINATION AND LONGITUDE OF THE LINE OF NODES.

By R. TOWNSEND.

THE following elementary but rigorous investigation of the formulæ for the variations of the inclination and longitude of the line of nodes may be acceptable to the younger class of students entering upon the subject of the planetary perturbations.

Round the sun, as centre, let a sphere of unit radius be conceived described intersecting the fixed plane of reference in the great circle aa; that of the momentary orbit of the



planet in the circle xy; and the plane perpendicular to the momentary line of nodes in the circle bb; let p be the position of the planet with respect to this sphere at the moment in question; pq the component of its angular motion in the plane of its orbit during the time dt; and qq' the perpendicular component subtending the small distance to which it is forced out of that plane by the disturbing action during the same interval. Then, bisecting the small arc pq at o,

connecting $q'o$, and producing the connecting great circle to meet aa and bb at x' and y' respectively, the plane of the great circle $x'oy'$ will be that of the new instantaneous orbit at the end of the time dt, and xx' and yy' will be the momentary changes in the longitude of the line of nodes and in the inclination of the orbit which it is our object to calculate in terms of the ordinary quantities usually employed for the purpose.

Letting fall the small perpendicular $x'x''$ from x' on xy, and denoting by ϕ the argument of latitude or the arc ox, we have

$$di = yy' \text{ and } d\omega = xx' = \frac{x'x''}{\sin i};$$

but, from the proportions

$$qq' : x'x'' :: \tfrac{1}{2}pq : \sin\phi \quad \text{and} \quad qq' : yy' :: \tfrac{1}{2}pq : \cos\phi,$$

we have $\quad yy' = \dfrac{2qq'}{pq} . \cos\phi \quad$ and $\quad x'x'' = \dfrac{2qq'}{pq} . \sin\phi.$

Let P be the perpendicular component of the disturbing force, that which alone causes the plane of the orbit to change position; then, since from its action alone continued during the time dt, the body has been forced out of that plane to a distance $= r.qq'$, r being the radius vector of the planet, we have

$$P = \frac{2r.qq'}{dt^2}, \text{ and therefore } 2qq' = \frac{Pdt^2}{r};$$

so that $\quad yy' = \dfrac{P.dt^2}{r.pq} . \cos\phi \quad$ and $\quad x'x'' = \dfrac{P.dt^2}{r.pq} . \sin\phi;$

but $r^2.pq = Hdt$, H being the momentary elementary area of the planet in the plane of its orbit; hence, substituting from this for pq, and dividing by dt, we have finally

$$\frac{di}{dt} = \frac{P}{H} . r\sin\phi \quad \text{and} \quad \frac{d\omega}{dt} = \frac{P}{H} . \frac{r\cos\phi}{\sin i} \quad \ldots\ldots (1),$$

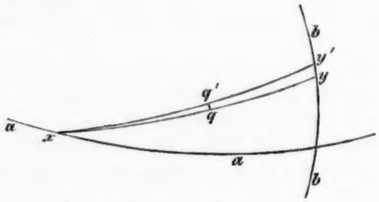
the most general expressions for the required variations, whatever be the inclination of the orbit or the nature of the curve described by the body.

But these formulæ not being practically applicable in their present form, as containing explicitly the actual position of the planet, the force P may be easily expressed in terms of the partial differential coefficient of the ordinary disturbing function R with respect to either of the variables i or ω, and

thus the simplest form of the ordinary formulæ may be readily obtained.

Let R and $R + dR$ be the values of the disturbing function for the positions q and q' respectively; then, from the characteristic property of that function, the disturbing force along the line joining those positions, that is the force P, $= -\dfrac{dR}{r.qq'}$, the distance $r.qq'$ being considered positive, the actual direction of the force being of course from q towards q', and R being supposed to have the same sign as in Airy and Pratt.

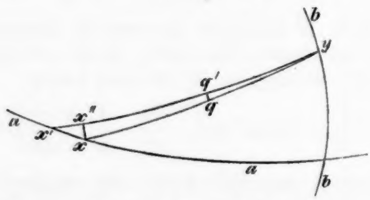
Now the change of position from q to q' might be conceived to take place by either of the two variables i or ω receiving a small change, the other in common with all the remaining elements of the orbit remaining unaltered, and we should have



In one case, i alone varying, (fig. 2)

$$qq' = yy_1 . \sin\phi = di . \sin\phi,$$

and therefore $$P = -\frac{1}{r\sin\phi} . \frac{dR}{di};$$



and in the other, ω alone varying, (fig. 3)

$$qq' = xx_1 . \sin i . \cos\phi = - d\omega . \sin i . \cos\phi,$$

and therefore $$P = \frac{1}{r\sin i . \cos\phi} . \frac{dR}{d\omega}.$$

Substituting the second and first of these values of P in the

first and second of formulæ (1) respectively, we get

$$\frac{di}{dt} = \frac{1}{H \sin i} \cdot \frac{dR}{d\omega} \quad \text{and} \quad \frac{d\omega}{dt} = -\frac{1}{H \sin i} \cdot \frac{dR}{di} \; \ldots\ldots (2);$$

the ordinary formulæ in their simplest form expressed like those for all the other variations in terms of the elements of the instantaneous orbit, partial differential coefficients of R with respect to the elements, and the time.

In an excellent article on the present subject by Mr. Blackburne, in the first volume of this *Journal* (new series), the formulæ (1) and (2) are obtained in a different way, and some important remarks are added in explanation of the proper method of applying them. See pp. 37 to 45.

Trinity College, Dublin,
Dec. 31, 1852.

ON DEFINITE INTEGRALS SUGGESTED BY THE THEORY OF HEAT.

By W. H. L. Russell, Esq., B.A., Shepperton, Middlesex.

Poisson has considered in the tenth chapter of his *Théorie de la Chaleur,* the case of the distribution of heat in a sphere of very great radius situated in a medium of variable temperature. In the course of his investigation he remarks that a certain part of the temperature of the interior of the sphere arising from a supposed increment of the exterior temperature vanishes with the time, and thus finds

$$\int_0^\infty \frac{\varepsilon^{-x\sqrt{\frac{1}{2}\alpha}}\left\{(p+\sqrt{\tfrac{1}{2}\alpha})\cos x\sqrt{\tfrac{1}{2}\alpha} - \sqrt{\tfrac{1}{2}\alpha}\sin x\sqrt{\tfrac{1}{2}\alpha}\right\} d\alpha}{(\alpha'^2-\alpha^2)(\alpha+p\sqrt{2\alpha}+p^2)}$$

$$= \frac{\pi\varepsilon^{-x\sqrt{\frac{1}{2}\alpha'}}\left\{(p+\sqrt{\tfrac{1}{2}\alpha'})\sin x\sqrt{\tfrac{1}{2}\alpha'} + \sqrt{\tfrac{1}{2}\alpha'}\cos x\sqrt{\tfrac{1}{2}\alpha'}\right\}}{2\alpha'(\alpha'+p\sqrt{2\alpha'}+p^2)}.$$

He then says, that this process "nous fait connaitre la valeur d'une integrale definie que l'on n'obtiendrait par aucun procedé direct, mais qui n'en est pas moins certaine, puis quelle est une consequence necessaire de notre analyse."

Now I am going to shew that this is not the case, but that the integral can be obtained without any reference to the Theory of Heat and by a direct process. I shall also investigate the values of some other integrals suggested by this process.

Let $u = \int_0^\infty \frac{\varepsilon^{-x\sqrt{\alpha}} \cos x\sqrt{\alpha}\, d\alpha}{\alpha'^2 - \alpha^2}$, whence

$$\frac{d^4u}{dx^4} + 4\alpha'^2 u = 4\int_0^\infty \varepsilon^{-x\sqrt{\alpha}} \cos x\sqrt{\alpha}\, d\alpha = 0,$$

therefore

$$u = C_1\varepsilon^{-x\sqrt{\alpha'}} \cos x\sqrt{\alpha'} + C_2\varepsilon^{-x\sqrt{\alpha'}} \sin x\sqrt{\alpha'} + C_3\varepsilon^{x\sqrt{\alpha'}} \cos x\sqrt{\alpha'} + C_4\varepsilon^{x\sqrt{\alpha'}} \sin x\sqrt{\alpha'}.$$

Now the integral cannot go on perpetually increasing with x, consequently $C_3 = C_4 = 0$. It remains to determine C_1 and C_2.

Let $x = 0$, then $C_1 = \int_0^\infty \frac{d\alpha}{\alpha'^2 - \alpha^2} = 0.$

Differentiate with regard to (x) and put $x = 0$, then we have

$$-\int_0^\infty \frac{d\alpha\sqrt{\alpha}}{\alpha'^2 - \alpha^2} = -\int_0^\infty \frac{d\sqrt{\alpha}}{\alpha' - \alpha} + \int_0^\infty \frac{d\sqrt{\alpha}}{\alpha' + \alpha} = C_2\sqrt{\alpha'}, \ \therefore\ C_2 = \frac{\pi}{2\alpha'}.$$

Hence we have

$$\int_0^\infty \frac{\varepsilon^{-x\sqrt{\alpha}} \cos x\sqrt{\alpha}\, d\alpha}{\alpha'^2 - \alpha^2} = \frac{\pi}{2\alpha'}\varepsilon^{-x\sqrt{\alpha'}} \sin x\sqrt{\alpha'}.$$

Put $\frac{1}{2}\alpha$ for α and $\frac{1}{2}\alpha'$ for α', and multiply by ε^{-px}, and we find

$$\int_0^\infty \frac{\varepsilon^{-x(p+\sqrt{\frac{1}{2}\alpha})} \cos x\sqrt{\tfrac{1}{2}\alpha}\, d\alpha}{\alpha'^2 - \alpha^2} = \frac{\pi}{2\alpha'}\varepsilon^{-x(p+\sqrt{\frac{1}{2}\alpha'})} \sin x\sqrt{\tfrac{1}{2}\alpha'}.$$

Hence, integrating with respect to x, and dividing by ε^{-px},

$$\int_0^\infty \frac{\varepsilon^{-x\sqrt{\frac{1}{2}\alpha}}\left\{(p+\sqrt{\tfrac{1}{2}\alpha})\cos x\sqrt{\tfrac{1}{2}\alpha} - \sqrt{\tfrac{1}{2}\alpha}\sin x\sqrt{\tfrac{1}{2}\alpha}\right\} d\alpha}{(\alpha'^2 - \alpha^2)(\alpha + p\sqrt{2\alpha} + p^2)}$$

$$= \frac{\pi\varepsilon^{-x\sqrt{\frac{1}{2}\alpha'}}\left\{(p+\sqrt{\tfrac{1}{2}\alpha'})\sin x\sqrt{\tfrac{1}{2}\alpha'} + \sqrt{\tfrac{1}{2}\alpha'}\cos x\sqrt{\tfrac{1}{2}\alpha'}\right\}}{2\alpha'(\alpha' + p\sqrt{2\alpha'} + p^2)}.$$

Again, to find $u = \int_0^\infty \frac{\varepsilon^{-x\sqrt{\alpha}} \cos x\sqrt{\alpha}\, d\alpha}{\alpha'^4 + \alpha^4}$, we have

$$u = \varepsilon^{-x\sqrt{\alpha'}\frac{\sqrt{(2+\sqrt{2})}}{\sqrt{2}}}\left\{C_1 \cos x\sqrt{\alpha'}\,\frac{\sqrt{(2-\sqrt{2})}}{\sqrt{2}} + C_2 \sin x\sqrt{\alpha'}\,\frac{\sqrt{(2-\sqrt{2})}}{\sqrt{2}}\right\}$$

$$+ \varepsilon^{-x\sqrt{\alpha'}\frac{\sqrt{(2-\sqrt{2})}}{\sqrt{2}}}\left\{C_3 \cos x\sqrt{\alpha'}\,\frac{\sqrt{(2+\sqrt{2})}}{\sqrt{2}} + C_4 \sin x\sqrt{\alpha'}\,\frac{\sqrt{(2+\sqrt{2})}}{\sqrt{2}}\right\},$$

omitting the terms which continually increase with (x). We find also

$$C_1 = C_2 = \frac{\pi}{2\alpha'^3\sqrt{2}},\quad C_3 = C_4 = 0;$$

therefore

$$\int_0^\infty \frac{\varepsilon^{-x\sqrt{\alpha}} \cos x\sqrt{\alpha}\, d\alpha}{\alpha'^4 + \alpha^4}$$

$$= \frac{\pi}{2\alpha'^3\sqrt{2}}\, \varepsilon^{-x\sqrt{\alpha'}\frac{\sqrt{(2+\sqrt{2})}}{\sqrt{2}}} \left\{\sin x\sqrt{\alpha'}\,\frac{\sqrt{(2-\sqrt{2})}}{\sqrt{2}} + \cos x\sqrt{\alpha'}\,\frac{\sqrt{(2-\sqrt{2})}}{\sqrt{2}}\right\}.$$

If $u = \int_0^\infty \frac{\varepsilon^{-x\sqrt{\alpha}} \cos x\sqrt{\alpha}\, d\alpha}{\alpha'^2 + \alpha^2}$, we find $\frac{d^4u}{dx^4} - 4\alpha'^2 u = 0$,

therefore $u = C_1\varepsilon^{-x\sqrt{2\alpha'}} + C_2\varepsilon^{x\sqrt{2\alpha'}} + C_3 \cos x\sqrt{2\alpha'} + C_4 \sin x\sqrt{2\alpha'}$.

The integral must continually decrease as x increases, therefore $C_2 = C_3 = C_4 = 0$.

If we put $x = 0$, we have

$$C_1 = \int_0^\infty \frac{d\alpha}{\alpha'^2 + \alpha^2} = \frac{\pi}{2\alpha'},$$

therefore $$\int_0^\infty \frac{\varepsilon^{-x\sqrt{\alpha}} \cos x\sqrt{\alpha}\, d\alpha}{\alpha'^2 + \alpha^2} = \frac{\pi}{2\alpha'}\,\varepsilon^{-x\sqrt{2\alpha'}}.$$

Put $\frac{1}{2}\alpha$ for α, and $\frac{1}{2}\alpha'$ for α', and multiply by ε^{-px}, then

$$\int_0^\infty \frac{\varepsilon^{-(p+\sqrt{\frac{1}{2}\alpha})x} \cos x\sqrt{\alpha}\, d\alpha}{\alpha'^2 + \alpha^2} = \frac{\pi}{2\alpha'}\,\varepsilon^{-(p+\sqrt{\alpha'})x},$$

$$\therefore \int_0^\infty \frac{\varepsilon^{-x\sqrt{\frac{1}{2}\alpha}}\{(p+\sqrt{\tfrac{1}{2}\alpha}) \cos x\sqrt{\tfrac{1}{2}\alpha} - \sqrt{\tfrac{1}{2}\alpha}\sin x\sqrt{\tfrac{1}{2}\alpha}\}}{(\alpha'^2+\alpha^2)(\alpha + p\sqrt{2\alpha} + p^2)} = \frac{\pi}{2\alpha'}\,\frac{\varepsilon^{-x\sqrt{\alpha'}}}{p+\sqrt{\alpha'}}.$$

NOTES ON MOLECULAR MECHANICS.

By the Rev. Samuel Haughton.

No. 2.—*Propagation of Plane Waves.*

The equations of small motions of elastic media are as follows:

$$\left.\begin{aligned}
-\varepsilon\frac{d^2\xi}{dt^2} &= \frac{d}{dx}\cdot\frac{dV}{d\alpha_1} + \frac{d}{dy}\cdot\frac{dV}{d\alpha_2} + \frac{d}{dz}\cdot\frac{dV}{d\alpha_3}\\
-\varepsilon\frac{d^2\eta}{dt^2} &= \frac{d}{dx}\cdot\frac{dV}{d\beta_1} + \frac{d}{dy}\cdot\frac{dV}{d\beta_2} + \frac{d}{dz}\cdot\frac{dV}{d\beta_3}\\
-\varepsilon\frac{d^2\zeta}{dt^2} &= \frac{d}{dx}\cdot\frac{dV}{d\gamma_1} + \frac{d}{dy}\cdot\frac{dV}{d\gamma_2} + \frac{d}{dz}\cdot\frac{dV}{d\gamma_3}
\end{aligned}\right\}\ldots (4).*$$

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* Vide vol. IV. p. 174 and p. 173, for the meaning of α_1, α_2, &c.

Let V be the most general function of the second order of the differential coefficients α_1, α_2, &c.; then

$$-2V = (a_1^2)\alpha_1^2 + (a_2^2)\alpha_2^2 + \&c. + (b_1^2)\beta_1^2 + \&c.$$
$$+ (a_1b_2)\alpha_1\beta_2 + \&c. \ (c_2a_3)\gamma_2\alpha_3 + \&c.$$

Introducing this value of V into equations (4), we find

$$\left.\begin{aligned}
-\varepsilon\frac{d^2\xi}{dt^2} = (a_1^2)\frac{d^2\xi}{dx^2} &+ (a_2^2)\frac{d^2\xi}{dy^2} + (a_3^2)\frac{d^2\xi}{dz^2} \\
&+ 2(a_2a_3)\frac{d^2\xi}{dy\,dz} + 2(a_1a_3)\frac{d^2\xi}{dx\,dz} + 2(a_1a_2)\frac{d\xi^2}{dx\,dy} \\
&+ (a_1b_1)\frac{d^2\eta}{dx^2} + (a_2b_2)\frac{d^2\eta}{dy^2} + (a_3b_3)\frac{d^2\eta}{dz^2} \\
+ (a_2b_3 + a_3b_2)\frac{d^2\eta}{dy\,dz} &+ (a_1b_3 + a_3b_1)\frac{d^2\eta}{dx\,dz} + (a_1b_2 + a_2b_1)\frac{d^2\eta}{dx\,dy} \\
&+ (a_1c_1)\frac{d^2\zeta}{dx^2} + (a_2c_2)\frac{d^2\zeta}{dy^2} + (a_3c_3)\frac{d^2\zeta}{dz^2} \\
+ (a_2c_3 + a_3c_2)\frac{d^2\zeta}{dy\,dz} &+ (a_1c_3 + a_3c_1)\frac{d^2\zeta}{dx\,dz} + (a_1c_2 + a_2c_1)\frac{d^2\zeta}{dx\,dy}
\end{aligned}\right\} \ldots\ldots (9),$$

$$-\varepsilon\frac{d^2\eta}{dt^2} = \&c.$$

$$-\varepsilon\frac{d^2\zeta}{dt^2} = \&c.$$

Let the following values of ξ, η, ζ, be introduced into equations (9),

$$\xi = \cos\alpha \times \text{function of } \frac{2\pi}{\lambda}(lx + my + nz - vt),$$
$$\eta = \cos\beta \times \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\zeta = \cos\gamma \times \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

in which l, m, n are the direction cosines of the wave normal and α, β, γ the direction angles of molecular vibration, which is supposed rectilinear. This particular integral corresponds to the case of rectilinear vibrations of plane waves, propagated without diminution of intensity.

We find from the substitution

$$\left.\begin{aligned}
\varepsilon v^2\cos\alpha &= P'\cos\alpha + H'\cos\beta + G'\cos\gamma \\
\varepsilon v^2\cos\beta &= Q'\cos\beta + F'\cos\gamma + H'\cos\alpha \\
\varepsilon v^2\cos\gamma &= R'\cos\gamma + G'\cos\alpha + F'\cos\beta
\end{aligned}\right\} \ldots (10),$$

in which

$$P' = (a_1^2)l^2 + (a_2^2)m^2 + (a_3^2)n^2 + 2(a_2a_3)mn + 2(a_1a_3)ln + 2(a_1a_2)lm,$$

$$Q' = (b_1^2)l^2 + (b_2^2)m^2 + (b_3^2)n^2 + 2(b_2b_3)mn + 2(b_1b_3)ln + 2(b_1b_2)lm,$$

$$R' = (c_1^2)l^2 + (c_2^2)m^2 + (c_3^2)n^2 + 2(c_2c_3)mn + 2(c_1c_3)ln + 2(c_1c_2)lm,$$

$$F' = (b_1c_1)l^2 + (b_2c_2)m^2 + (b_3c_3)n^2 + (b_2c_3 + b_3c_2)mn + (b_1c_3 + b_3c_1)ln + (b_1c_2 + b_2c_1)lm,$$

$$G' = (a_1c_1)l^2 + (a_2c_2)m^2 + (a_3c_3)n^2 + (a_2c_3 + a_3c_2)mn + (a_1c_3 + a_3c_1)ln + (a_1c_2 + a_2c_1)lm,$$

$$H' = (a_1b_1)l^2 + (a_2b_2)m^2 + (a_3b_3)n^2 + (a_2b_3 + a_3b_2)mn + (a_1b_3 + a_3b_1)ln + (a_1b_2 + a_2b_1)lm.$$

Equations (10) are the well-known equations which determine the axes of the ellipsoid

$$P'x^2 + Q'y^2 + R'z^2 + 2F'yz + 2G'xz + 2H'xy = 1 \ldots (11);$$

there are, therefore, three possible directions of molecular vibration for a given direction of wave plane; and there will be three parallel waves moving with velocities determined by the magnitude of the axes of the ellipsoid, the direction of vibration in each plane wave being parallel to one of the axes.

The construction of the direction of molecular vibration just found was given by M. Cauchy* for a system of attracting and repelling molecules; it is here shewn to be a necessary consequence of the assumption of a function such as V, to represent the effects of molecular force. Hence we may state the following theorem:

If the sum of the molecular moments of an elastic body can be represented by the variation of a single function, the directions of molecular vibration, corresponding to a given direction of wave plane, must be at right angles to each other.

The converse of this theorem in molecular mechanics has been proved by Professor Jellett.†

The cubic equation whose roots are the squares of the reciprocals of the axes of the ellipsoid (11) is

$$(P' - s)(Q' - s)(R' - s) - F'^2(P' - s) - G'^2(Q' - s) - H'^2(R' - s) + 2F'G'H' = 0,$$

where $s = \varepsilon v^2$; hence, if P, Q, R, F, G, H be the same

* *Exercises des Mathematiques*, tom. v. p. 32.

† *Transactions of Royal Irish Academy*, vol. XXII. p. 195.

functions of x, y, z, that P', Q', R', F', G', H' are of l, m, n; it can easily be shewn that the surface of *wave-slowness* is

$$(P-1)(Q-1)(R-1) - F^2(P-1) - G^2(Q-1) - H^2(R-1) + 2FGH = 0 \ldots\ldots(12.).$$

It is natural to inquire whether a series of plane waves may not be propagated, with a continually decreasing amplitude of vibration: to ascertain the possibility of this, let us assume

$$\left.\begin{aligned} \xi &= p\cos\alpha \,.\, e^{-\frac{2\pi}{\lambda}\rho(lx+my+nz)} \sin\frac{2\pi}{\lambda}(lx+my+nz-vt) \\ \eta &= p\cos\beta \text{ \&c.} \\ \zeta &= p\cos\gamma \text{ \&c.} \end{aligned}\right\} \ldots(13).$$

This is equivalent to assuming that the amplitude of vibration diminishes as the plane wave moves away from the plane $lx + my + nz = 0$, passing through the origin of coordinates; in fact, $lx + my + nz$ is equal to the perpendicular let fall from the point (x, y, z) upon that plane, and the vibration is here assumed to diminish in geometrical progression, as that perpendicular increases in arithmetical progression.

In order to simplify the introduction of (ξ, η, ζ) into the differential equations, we may use the following symbolical equation,

$$e^{-q}\sin p = \sin\{p + q\sqrt{(-1)}\} \ldots\ldots\ldots(14).$$

This is equivalent to Fresnel's interpretation of imaginary quantities, and assumes that

$$\sqrt{(-1)}\sin f = \sin(f + \tfrac{1}{2}\pi),$$
$$\sqrt{(-1)}\cos f = \cos(f + \tfrac{1}{2}\pi).$$

Introducing, by the aid of (14), or by direct differentiation, the values of ξ, η, ζ, into equations (9), we find

$$\left.\begin{aligned} &\frac{\varepsilon v^2}{1-\rho^2}\cos\alpha = P'\cos\alpha + H'\cos\beta + G'\cos\gamma \\ &\frac{\varepsilon v^2}{1-\rho^2}\cos\beta = Q'\cos\beta + F'\cos\gamma + H'\cos\alpha \\ &\frac{\varepsilon v^2}{1-\rho^2}\cos\gamma = R'\cos\gamma + G'\cos\alpha + F'\cos\beta \\ \text{and}\quad &2\rho\sqrt{(-1)}\,(P'\cos\alpha + H'\cos\beta + G'\cos\gamma) = 0, \\ &2\rho\sqrt{(-1)}\,(Q'\cos\beta + \text{\&c.}) \ldots\ldots\ldots\ldots\ldots\ldots = 0, \\ &2\rho\sqrt{(-1)}\,(R'\cos\gamma + \text{\&c.}) \ldots\ldots\ldots\ldots\ldots\ldots = 0, \end{aligned}\right\} \ldots(15),$$

giving six equations of condition: but these equations cannot coexist unless $\rho = 0$, *i.e.* unless there be no diminution of intensity.

If we assume

$$\left.\begin{aligned} \xi &= p\cos\alpha . e^{-\frac{2\pi}{\lambda}\{\rho(lx+my+nz)-Kvt\}} \sin\frac{2\pi}{\lambda}(lz+my+nz-vt) \\ \eta &= \&c. \\ \zeta &= \&c. \end{aligned}\right\} \ldots(16),$$

so as to introduce another unknown constant K, we should find for the equations of condition

$$\left.\begin{aligned} \frac{1-K^2}{1-\rho^2}\,\varepsilon v^2 \cos\alpha &= P'\cos\alpha + H'\cos\beta + G'\cos\gamma \\ \frac{1-K^2}{1-\rho^2}\,\varepsilon v^2 \cos\beta &= Q'\cos\beta + \&c. \\ \frac{1-K^2}{1-\rho^2}\,\varepsilon v^2 \cos\gamma &= R'\cos\gamma + \&c. \\ \text{and}\quad \frac{K}{\rho}\,\varepsilon v^2 \cos\alpha &= P'\cos\alpha + H'\cos\beta + G'\cos\gamma \\ \frac{K}{\rho}\,\varepsilon v^2 \cos\beta &= Q'\cos\beta + \&c. \\ \frac{K}{\rho}\,\varepsilon v^2 \cos\gamma &= R'\cos\gamma + \&c. \end{aligned}\right\} \ldots(17).$$

These equations of condition can only coexist on the supposition that

$$\frac{1-K^2}{1-\rho^2} = \frac{K}{\rho} \ldots\ldots\ldots\ldots\ldots(18).$$

Equation (18), which is quadratic, if solved for K, will give

$$K_1 = \rho, \quad K_2 = -\frac{1}{\rho};$$

the latter is to be rejected, as it would give an imaginary velocity, and the former introduced into the expression (16) gives, if $\phi = \frac{2\pi}{\lambda}(lx + my + nz - vt)$,

$$\left.\begin{aligned} \xi &= p\cos\alpha\, e^{-\rho\phi} \sin\phi \\ \eta &= p\cos\beta\, e^{-\rho\phi} \sin\phi \\ \zeta &= p\cos\gamma\, e^{-\rho\phi} \sin\phi \end{aligned}\right\} \ldots\ldots (16) \text{ bis.}$$

These values of ξ, η, ζ are functions of the phase, but do not indicate a diminished intensity of vibration: this agrees with a statement of Mr. Rankine's, *Math. Jour.*, vol. VII. p. 218.

It is desirable, before concluding this note, to give some idea of the true nature of the differential equations which have been used. We have assumed that the density is very little altered during the motion of the particles: if we take account of the alteration of density, the equations will be different.

The equation of continuity is

$$\frac{d\varepsilon}{dt} + \frac{d}{dx}(\varepsilon u) + \frac{d}{dy}(\varepsilon v) + \frac{d}{dz}(\varepsilon w) = 0 \;\ldots\; (19),$$

where u, v, w, denote the velocities parallel to x, y, z. For small displacements this will become

$$\frac{d\varepsilon}{dt} + \varepsilon\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right),$$

or

$$\frac{d\varepsilon}{dt} + \varepsilon\frac{d\omega}{dt} = 0 \;\ldots\ldots\ldots\ldots\; (20),$$

in which $\omega = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}$ = cubical compression.

Integrating (20), we obtain

$$\log\varepsilon + \omega = F(x, y, z),$$

which gives, since the cubical compression is supposed zero before the motion of the wave reaches a particle,

$$\varepsilon = \varepsilon_0 e^{-\omega} \;\ldots\ldots\ldots\ldots\ldots\; (21),$$

ε_0 denoting the statical density of the medium, and e denoting the napierian base.

Introducing this value of the density into equations (4), we find

$$\left.\begin{aligned}
-\varepsilon_0 e^{-\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)}\frac{d^2\xi}{dt^2} &= \frac{d}{dx}\frac{dV}{d\alpha_1} + \frac{d}{dy}\frac{dV}{d\alpha_2} + \frac{d}{dz}\frac{dV}{d\alpha_3}\\
-\varepsilon_0 e^{-\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)}\frac{d^2\eta}{dt^2} &= \frac{d}{dx}\frac{dV}{d\beta_1} + \frac{d}{dy}\frac{dV}{d\beta_2} + \frac{d}{dz}\frac{dV}{d\beta_3}\\
-\varepsilon_0 e^{-\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)}\frac{d^2\zeta}{dt^2} &= \frac{d}{dx}\frac{dV}{d\gamma_1} + \frac{d}{dy}\frac{dV}{d\gamma_2} + \frac{d}{dz}\frac{dV}{d\gamma_3}
\end{aligned}\right\}\ldots(22);$$

we must, therefore, inquire into the limitation involved in the use of equations (4). If the medium were absolutely

incompressible, the equation of continuity would become

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0,$$

and equations (22) would coincide with (4); but the medium may be nearly incompressible and equations (4) used. In the case of plane waves the limitation may be thus formed,

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = \frac{2\pi}{\lambda} p (l \cos\alpha + m \cos\beta + n \cos\gamma) \cos\phi ;$$

or, if ϑ be the angle between the wave normal and direction of vibration, equation (21) will be

$$\varepsilon = \varepsilon_0 e^{-\frac{2\pi}{\lambda} p \cos\vartheta \cos\phi} \quad \text{..........} \ (23).$$

In order that ε shall not differ much from ε_0, it is, therefore, necessary to suppose that the vibration is nearly transversal, in which case $\cos\vartheta$ is very small; or, that if the vibration be nearly normal, λ must be very great; *i.e.* the velocity of the normal vibration must be very great.

In my next note I shall consider the case of vibrations, rigorously normal and transversal.

Trinity College, Dublin,
Dec. 30, 1852.

THEOREMS IN THE CALCULUS OF OPERATIONS.

By Robert Carmichael.

Those who have studied the Calculus of Operations in connexion with the Integral Calculus cannot but have felt a difficulty in the interpretation of the symbolic results at which they may have arrived. The farther the relation between these two subjects is prosecuted, whether in the solution of Differential Equations, the extension of Definite Integrals, or the reduction of equations in Finite Differences, the more imperative becomes the demand for such interpretation. In all these cases, so long as the solutions are symbolic and not completely evaluated, they are unsatisfactory to the advanced mathematician and perhaps calculated to lead the younger student to undervalue the utility of prosecuting these branches of analysis in conjunction.

The want here indicated can only be satisfied by the contributions of individuals. Much still is required, although

much has already been done. The present paper, with two others previously published in this *Journal*, are offered by me in furtherance of the object.

There is another matter to which it may be well to direct attention. When the questions to be investigated have a symmetrical character, not only should the results be symmetrical, but symmetrical methods should be employed for their deduction. A regard to this latter point might possibly have not only precluded some errors and many incomplete results, but also led the way to the discovery of higher and more elegant methods of analysis.

In conclusion, I would express my acknowledgments to the Rev. Professor Graves, without whose valuable suggestions these pages would never have been written. I have done little more than generalize and apply results communicated by him to the Royal Irish Academy in the month of April, 1852, an abstract of which is published in the Proceedings of that body.

1. Let it be proposed to investigate the value of the symbolic quantity

$$e^{\phi(x)\frac{d}{dx} + \psi(y)\frac{d}{dy} + \chi(z)\frac{d}{dz} + \&c.}\, U,$$

where $$U = f(x, y, z, \&c.).$$

Now if we put

$$\frac{dx}{\phi(x)} = d\xi, \quad \frac{dy}{\psi(y)} = d\eta, \quad \frac{dz}{\chi(z)} = d\zeta, \ \&c. \ldots\ldots (1),$$

this becomes

$$e^{\frac{d}{d\xi} + \frac{d}{d\eta} + \frac{d}{d\zeta} + \&c.}\, U;$$

and as U, from being a function of x, y, z, &c., can be transformed into a function of ξ, η, ζ, &c., by the aid of (1), we have reduced the question to a shape which admits of obvious solution. Thus, as

$$U = f(x, y, z, \&c.),$$

and

$$\left.\begin{aligned} \xi + c &= \int \frac{dx}{\phi(x)} = \Phi(x) \\ \eta + d &= \int \frac{dy}{\psi(y)} = \Psi(y) \\ \zeta + e &= \int \frac{dz}{\chi(z)} = \mathrm{X}(z) \\ &\&c. \end{aligned}\right\}$$

we get

$$U = f\{\Phi^{-1}(\xi + c),\ \Psi^{-1}(\eta + d),\ \mathrm{X}^{-1}(\zeta + e),\ \&c.\},$$

and therefore

$$e^{\frac{d}{d\xi} + \frac{d}{d\eta} + \&c.} U = f\{\Phi^{-1}(\xi + c + 1),\ \Psi^{-1}(\eta + d + 1),\ \&c.\};$$

whence, finally,

$$e^{\phi(x)\frac{d}{dx} + \psi(y)\frac{d}{dy} + \&c.} f(x, y, \&c.) = f\{\Phi^{-1}(\Phi x + 1), \Psi^{-1}(\Psi y + 1), \&c.\}.$$

In the practical application of this fundamental theorem the only difficulties with which we have to contend are, the deduction of the integrals

$$\int \frac{dx}{\phi(x)},\quad \frac{dy}{\psi(y)},\quad \&c.,$$

and the inversion of the functions Φ, Ψ, &c.

Ex. I. The simplest and most obvious illustration of this theorem is afforded by the suppositions

$$\phi(x) = x,\quad \psi(y) = y,\quad \chi(z) = z,\quad \&c.$$

In this case, in fact, the operative symbol

$$\phi(x)\frac{d}{dx} + \psi(y)\frac{d}{dy} + \chi(z)\frac{d}{dz} + \&c.$$

becomes the index symbol of homogeneous functions

$$x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz} + \&c. = \nabla,$$

and therefore

$$e^{\nabla}.f(x, y, \&c.) = f\{\log^{-1}(1 + \log x),\ \log^{-1}(1 + \log y),\ \&c.\},$$

or $\qquad e^{\nabla}.f(x, y, z, \&c.) = f(ex, ey, ez, \&c.).$

If we break up f into sets of homogeneous terms, it is evident that this result is identical with that given in a previous paper, namely

$$e^{\nabla} U = u_0 + eu_1 + e^2 u_2 + \&c. + e^n u_n.$$

Ex. II. More generally, let

$$\phi(x) = x^m,\quad \psi(y) = y^n,\quad \&c.,$$

and the result of the evaluation of

$$e^{x^m \frac{d}{dx} + y^n \frac{d}{dy} + \&c.} f(x, y, \&c.)$$

is

$$f\left[\frac{x}{\{1-(m-1)x^{m-1}\}^{\frac{1}{m-1}}},\ \frac{y}{\{1-(n-1)y^{n-1}\}^{\frac{1}{n-1}}},\ \&c.\right].$$

It is not difficult to verify this formula for the particular cases

$$m = n = \&c. = 0,$$

$$m = n = \&c. = 1.$$

In that in which

$$m = n = \&c. = 2,$$

we get the result

$$e^{x^2\frac{d}{dx}+y^2\frac{d}{dy}+\&c.}f(x, y, \&c.) = f\left(\frac{x}{1-x},\ \frac{y}{1-y},\ \&c.\right).$$

Ex. III. Let

$$\phi(x) = (a^2 - x^2)^{\frac{1}{2}},\quad \psi(y) = (b^2 - y^2)^{\frac{1}{2}},\quad \&c.$$

and the result of the evaluation of

$$e^{(a^2-x^2)^{\frac{1}{2}}\frac{d}{dx}+(b^2-y^2)^{\frac{1}{2}}\frac{d}{dy}+\&c.}f(x, y, \&c.)$$

is

$$f\{\sin(1 + \sin^{-1}x),\ \sin(1 + \sin^{-1}y),\ \&c.\}.$$

Ex. IV. Selecting now a particular form for the function operated on, we shall suppose, the simplest case, that it is linear in x, y, &c. Then

$$e^{\phi(x)\frac{d}{dx}+\psi(y)\frac{d}{dy}+\&c.}(ax + by + \&c.)$$

$$= a\Phi^{-1}(\Phi x + 1) + b\Psi^{-1}(\Psi y + 1) + \&c.,$$

and we may introduce the values of $\phi(x)$, $\psi(y)$, &c., employed in the previous examples.

There are certain cases in which the evaluation of the quantity

$$e^{\phi(x)\frac{d}{dx}+\psi(y)\frac{d}{dy}+\&c.}U$$

may be considerably facilitated. Thus, if it consist of the product of certain minor functions u, v, w, &c., we may avail ourselves of the theorem given in the last number of this *Journal*, namely that

$$e^{\psi}.uvw\ldots = e^{\psi}u.e^{\psi}v.e^{\psi}w\ldots$$

if ψ be such a symbol that

$$\psi.uv = u\psi v + v\psi u;$$

since it is obvious that

$$\phi(x)\frac{d}{dx}+\psi(y)\frac{d}{dy}+\&c.$$

satisfies the required condition.

2. If we operate on both sides of the fundamental theorem with

$$e^{\phi x\frac{d}{dx}+\psi y\frac{d}{dy}+\&c.}$$

we easily find that

$$e^{2\left(\phi x\frac{d}{dx}+\psi y\frac{d}{dy}+\&c.\right)}f(x,y,\&c.)=f\{\Phi^{-1}(\Phi x+2),\ \Psi^{-1}(\Psi y+2),\ \&c.\};$$

and hence, in general, that

$$e^{m\left(\phi x\frac{d}{dx}+\psi y\frac{d}{dy}+\&c.\right)}f(x,y,\&c.)=f\{\Phi^{-1}(\Phi x+m),\Psi^{-1}(\Psi y+m),\ \&c.\}$$

Thus, the form of f being supposed unknown, and those of Φ and Ψ given, the solution of the equation of finite differences, with constant coefficients,

$$\left.\begin{array}{c} Af\{\Phi^{-1}(\Phi x+a),\ \ \Psi^{-1}(\Psi y+a)\} \\ + \\ Bf\{\Phi^{-1}(\Phi x+b),\ \ \Psi^{-1}(\Psi y+b)\} \\ +\ \&c. \end{array}\right\}=0,$$

is reduced to the solution of the symbolic partial differential equation

$$Ae^{a\left(\phi x\frac{d}{dx}+\psi y\frac{d}{dy}\right)}z+Be^{b\left(\phi x\frac{d}{dx}+\psi y\frac{d}{dy}\right)}z+\&c.=0,$$

which may be written, for brevity,

$$F(e^{\phi x\frac{d}{dx}+\psi y\frac{d}{dy}})\,z=0\ ;$$

or, by the previous transformation,

$$F(e^{\frac{d}{d\xi}+\frac{d}{d\eta}})\,z=0.$$

Now, if the roots of $\quad F(p)=0$

be all real and unequal, the symbolic solution of this equation is

$$z=(e^{\frac{d}{d\xi}+\frac{d}{d\eta}}-m)^{-1}.\,0+(e^{\frac{d}{d\xi}+\frac{d}{d\eta}}-n)^{-1}.\,0+\&c.,$$

where m, n, &c. are the values of the roots.

But by a previous theorem (*Journal*, 1851)

$$\chi\left(\frac{d}{d\xi}+\frac{d}{d\eta}\right).f_m(e^{\xi}, e^{\eta}) = \chi(m).f_m(e^{\xi}, e^{\eta}) \ldots\ldots (2),$$

f_m being a homogeneous function of the m^{th} degree.

Hence the solution of the symbolic equation, and therefore the solution of the equation of finite differences is

$$z = u_{\log m}(e^{\xi}, e^{\eta}) + u_{\log n}(e^{\xi}, e^{\eta}) + \&c.,$$

where *the forms* of $u_{\log m}$, $u_{\log n}$, &c. are arbitrary, but their degrees given by the suffixes.

Finally, introducing the arbitrary constants c, d, &c., as is evidently legitimate, and then substituting their values for $\xi + c$, $\eta + d$, &c., we get the solution in the form

$$z = u_{\log m}(e^{\Phi x}, e^{\Psi y}) + u_{\log n}(e^{\Phi x}, e^{\Psi y}) + \&c.$$

If $$F(p) = 0$$

contain pairs of imaginary roots, the solution assumes the form

$$z = u_{\log(m+n\sqrt{-1})}(e^{\Phi x}, e^{\Psi y}) + u_{\log(m-n\sqrt{-1})}(e^{\Phi x}, e^{\Psi y}) + \&c. + u_{\log p} + \&c.$$

Finally, if the same equation contain α equal roots, whose common value is m, the form of the solution is

$$z = u_{\log m}(e^{\Phi x}, e^{\Psi y}).(\Phi x + \Psi y)^{\alpha-1} + v_{\log m}(e^{\Phi x}, e^{\Psi y}).(\Phi x + \Psi y)^{\alpha-2} + \&c.$$
$$+ u_{\log n}(e^{\Phi x}, e^{\Psi y}) + \&c.,$$

where $u_{\log m}$, $v_{\log m}$ are different arbitrary homogeneous functions of the same degree.

3. It is now at once obvious that we are prepared to solve such an equation in finite differences as

$$A\phi(x + a, y + a, \&c.) + B\phi(x + b, y + b, \&c.) + \&c. = 0,$$

either as an illustration of the previous article, or independently. Adopting the latter course, it is easy to see that we can solve the still higher equation

$$\Sigma A\phi(x + a, y + a, \&c.) = f(e^x, \sin x, e^y, \sin y, \&c.),$$

where we can reduce the right-hand member to the form

$$\Sigma A_{p,\,q,\,\&c.}\, e^{px + qy + \&c.},$$

p, q, &c. being positive or negative, integral or fractional, real or imaginary.

For, throwing the equation into the form

$$F(e^{\frac{d}{dx}+\frac{d}{dy}})\,\phi(x, y, \text{\&c.}) = \Sigma A_{p,\,q,\,\text{\&c.}}\, e^{px+qy+\text{\&c.}},$$

we have, by (2), the solution in the form

$$\phi = \Sigma A_{p,\,q,\,\text{\&c.}} \frac{e^{px+qy+\text{\&c.}}}{F(e^{p+q+\text{\&c.}})} + u_{\log m}(e^x, e^y, \text{\&c.}) + \text{\&c.},$$

the roots of $F(p) = 0$ being supposed all real and unequal.

It is evident that the solution of the equation in finite differences, in which there is but a single variable, is but a particular case of the form now stated.

Trinity College, Dublin,
March 1853.

ON THE RELATION BETWEEN THE VOLUME OF A TETRAHEDRON AND THE PRODUCT OF THE *sixteen* ALGEBRAICAL VALUES OF ITS SUPERFICIES.

By J. J. Sylvester, F.R.S.

The area of a triangle is related (as is well known) in a very simple manner to the 8 algebraical values of its perimeter: If we call the values of the squared sides of the triangle a, b, c, there will be nothing to distinguish the algebraical affections of sign of the simple lengths so as to entitle one to a preference over the other. The area of the triangle can only vanish by reason of the three vertices coming into a straight line; hence, according to the general doctrine of characteristics, we must have the Norm of

$$\sqrt{a} + \sqrt{b} + \sqrt{c},$$

containing as a factor some root or power of the expressions for the area of the triangle. The Norm in question being representable as $-N^2$ where N is the Norm of $a^{\frac{1}{2}} \pm b^{\frac{1}{2}} \pm c^{\frac{1}{2}}$, which is of 4 dimensions in the elements a, b, c, and undecomposable into rational factors, we infer that to a numerical factor *près* the square of the area must be identical with the Norm N, and thus, by a logical *coup-de-main*,

completely supersede all occasion for the ordinary geometrical demonstration given of this proposition, which in its turn, with certain superadded definitions, would admit of being adopted as the basis of an absolutely pure system of Analytical Trigonometry that should borrow nothing from the methods and results of sensuous or practical geometry. But into this speculation it is not my present purpose to enter: what I propose to do is to extend a similar mode of reasoning to space of three dimensions, and to point out a general theorem in determinants which is involved as a consequence in the generalization of the result of the inquiry when pushed forward into the regions of what may be termed Absolute or Universal Rational Space.

Let F, G, H, K be the four squared areas of the faces of a tetrahedron, and V the volume; then, since V only becomes zero in the case of the 4 vertices coming into the same plane, which is characterised by the equation

$$\sqrt{F} + \sqrt{G} + \sqrt{H} + \sqrt{K} = 0$$

subsisting, we infer that N the Norm of

$$\sqrt{F} \pm \sqrt{G} \pm \sqrt{H} \pm \sqrt{K}$$

must contain a power of V as a rational factor. V^2 is rational and of 3 dimensions in the squared edges; the Norm above spoken of is of 8 dimensions in the same. Consequently there is a rational factor, say Q, remaining, which is of 5 dimensions in the squared edges, and this factor I now proceed to determine, the other factor V^2 being, as is well known, a numerical product of the determinant

$$\begin{array}{ccccc} 0 & ab^2 & ac^2 & ad^2 & 1 \\ ba^2 & 0 & bc^2 & bd^2 & 1 \\ ca^2 & cb^2 & 0 & cd^2 & 1 \\ da^2 & db^2 & dc^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}$$

a, b, c, d being the 4 angular points of the tetrahedron. See *London and Edinburgh Phil. Mag.* 1852.

The quantity Q possesses an interest of a geometrical character; for if we call the radii of the 8 spheres which can be inscribed in a tetrahedron $r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8$, we evidently have $r_1 r_2 r_3 r_4 r_5 r_6 r_7 r_8 \times N = (3V)^8$. Hence ($R$), the product of the eight radii in question, $= \dfrac{3^8 V^8}{N} = \dfrac{3^8 V^6}{Q}$.

Consequently Q is the quantity which characterises the fact of one or more of *the radii of the inscribed spheres becoming infinite.* For the triangle there exists no corresponding property; this we know *à priori,* and can explain also analytically from the fact that if we call P the product of the radii of the 4 inscribable circles, ν the Norm of the perimeter, and A the area, we have

$$P\nu = 2^4A^4,$$

and
$$\nu = \frac{2^4A^4}{P} = A^2,$$

which contains no denominator capable of becoming zero, so that as long as the sides remain finite the curvature of the inscribed circles is incapable of vanishing.

To determine N as a function of the edges, and then to discover by actual division the value of $\frac{N}{V^2}$, would be the direct but an excessively tedious and almost impracticably difficult process. I have ever felt a preference for the *à priori* method of discovering forms whose properties are known, and never yet have met with an instance where analysis has denied to gentle solicitation conclusions which she would be loth to grant to the application of force. The case before us offers no exception to the truth of this remark. Q is a function of 5 dimensions in terms of the squared edges: let us begin by finding the value of that part of Q in which at most a certain set of 4 of these edges make their appearance, and to find which consequently the other 2 edges may be supposed zero without affecting the result. We may make two distinct hypotheses concerning these 2 edges; we may suppose that they are opposite, *i.e.* non-intersecting edges, or that they are contiguous, *i.e.* intersecting edges.

To meet the first hypothesis suppose $ab = 0$, $ce = 0$.

For convenience sake, use F, G, H, K to denote 16 times the square of each area, instead of the simple square of the areas. Call

$$16(abc)^2 = K,\quad 16(abd)^2 = H,\quad 16(acd)^2 = G,\quad 16(bcd)^2 = F.$$

Then $-K = (ab)^4+(ac)^4+(bc)^4-2(ab)^2(ac)^2-2(ab)^2(bc)^2-2(ac)^2(bc)^2$

$$= ac^4 + bc^4 - 2(ac)^2(bc)^2.$$

Similarly,
$$-H = ad^4 + bd^4 - 2(ad)^2(bd)^2,$$
$$-G = ca^4 + da^4 - 2ca^2\,da^2,$$
$$-F = cb^4 + db^4 - 2cb^2\,db^2.$$

Hence one value of $\sqrt{F} + \sqrt{G} + \sqrt{H} + \sqrt{K}$ will be

$$\sqrt{(-1)}\,\{(ac^2 - bc^2) + (bd^2 - ad^2) + (da^2 - ac^2) + (bc^2 - bd^2)\} = 0.$$

Hence, on this first supposition, the Norm vanishes. But V^2 does not vanish when $ab = 0$, $cd = 0$, for it becomes, *saving* a numerical factor,

$$\begin{array}{ccccc} 0 & 0 & ac^2 & ad^2 & 1 \\ 0 & 0 & bc^2 & bd^2 & 1 \\ ca^2 & cb^2 & 0 & 0 & 1 \\ da^2 & db^2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & \end{array}$$

i.e.
$$\begin{aligned} &(ac^2.bd^2 - ad^2.bc^2).(cb^2 + ad^2 - ca^2 - bd^2) \\ &\quad + (bc^2 - ac^2)\,(ca^2.db^2 - cb^2.da^2) \\ &\quad + (ad^2 - bd^2)\,(ca^2.db^2 - cb^2.da^2) \\ &= 2\,(ac^2.bd^2 - ad^2.bc^2).(ad^2 + bc^2 - ac^2 - bd^2); \end{aligned}$$

and consequently, since N vanishes but V^2 does not vanish, (Q) vanishes, shewing that there is no term in (Q) but what contains one at least of any two opposite edges as a factor; or, in other words, there is no term in Q of which the product of the square of the product of all three sides of some one or other of the 4 faces does not form a constituent part.

Next, let us suppose $ab = 0$, $ac = 0$, then

$$\begin{aligned} K^2 &= 16abc^2 = -\,bc^4, \\ H^2 &= 16abd^2 = -\,(ad^2 - bd^2)^2, \\ G^2 &= 16acd^2 = -\,(ad^2 - cd^2)^2, \\ F^2 &= 16bcd^2 = -\,bc^4 - bd^4 - cd^4 + 2bc^2.bd^2 + 2bc^2.cd^2 + 2bd^2.cd^2. \end{aligned}$$

Four of the four factors of N will be therefore

$$\{\iota(bc^2 + cd^2 - bd^2) \pm F\},\quad \{\iota(bc^2 - cd^2 + bd^2) \pm F\},$$

ι denoting $\sqrt{(-1)}$, and the product of these four factors will be

$$\{(bc^2 + cd^2 - bd^2)^2 + F^2\} \times \{(bc^2 - cd^2 + bd^2)^2 + F^2\},$$

which is equal to

$$16bc^4.bd^2.cd^2;$$

and similarly, the remaining part of the Norm will be

$$\{(2ad^2 - bd^2 - cd^2 + bc^2)^2 + F^2\}\,.\,\{(2ad^2 - bd^2 - cd^2 - bc^2)^2 + F^2\},$$

that is

$$\{4ad^4 - 4ad^2(bd^2 + cd^2 + bc^2) + 4bc^2.bd^2 + 4bd^2.cd^2 + 4cd^2.bc^2\}$$
$$\times \{4ad^4 - 4ad^2(bd^2 + cd^2 - bc^2) + 4bd^2.cd^2\}.$$

Again, since $ac^2 = 0$ and $bc^2 = 0$, V^2 becomes

$$\begin{array}{ccccc} 0 & 0 & 0 & ad^2 & 1 \\ 0 & 0 & bc^2 & bd^2 & 1 \\ 0 & cb^2 & 0 & cd^2 & 1 \\ da^2 & db^2 & dc^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}$$

This is evidently equal to

$$2bc^2\left\{\begin{array}{cccc} 0 & 0 & ad^2 & 1 \\ 0 & cb^2 & cd^2 & 1 \\ da^2 & db^2 & 0 & 1 \\ 1 & 1 & 1 & \end{array}\right\} - bc^4\left\{\begin{array}{ccc} 0 & ad^2 & 1 \\ da^2 & 1 & 1 \\ 1 & 0 & \end{array}\right\}.$$

(the arrows being used to denote the directions of the positive diagonal sets of terms)

$$= 2bc^2\{2bc^2ad^2 + ad^4 - ad^2bd^2 - cd^2ad^2 + bd^2cd^2\}$$
$$- 2bc^4ad^2$$
$$= 2bc^2\{ad^4 - ad^2(bd^2 + cd^2 - bc^2) + bd^2.cd^2\}.$$

Hence, paying no attention to any mere numerical factor, we have found that when $ac = 0$ and $bc = 0$, Q or $\dfrac{N}{V^2}$ becomes

$$bc^2.bd^2.cd^2\{ad^4 - ad^2(bd^2 + cd^2 + bc^2) + bc^2.bd^2 + bd^2.cd^2 + cd^2.bc^2\}.$$

Hence, with the exception of the terms in which 5 out of the six edges enter, the complete value of Q will be

$$\Sigma(bc^2.bd^2.cd^2)\{ad^4 - ad^2(bd^2 + cd^2 + bc^2) + bc^2.bd^2 + bd^2.cd^2 + cd^2.bc^2\},$$

or more fully expressed, and still abstracting from terms containing 5 edges,

$$= \Sigma bc^2.bd^2.cd^2\{(ab^4 + ac^4 + ad^4) - (ab^2 + ac^2 + bc^2)(bd^2 + bc^2 + cd^2)$$
$$+ bc^2.bd^2 + bd^2.cd^2 + cd^2.bc^2\}.$$

It remains only to determine the value of the numerical coefficient affecting each of the 6 terms of the form

$$ab^2.ac^2.ad^2.bc^2.bd^2.$$

To find this, let

$$ab^2 = ac^2 = ad^2 = bc^2 = bd^2 = cd^2 = 1\,;$$

then evidently, since all the squared areas are equal, several of the factors of N will become zero, but V^2 evidently does not become zero for a regular tetrahedron; hence Q becomes zero: and if we call the numerical factor sought for λ, we must have (observing that the Σ includes 4 parts corresponding to each of the 4 faces)

$$4\{3-9+3\}+6\lambda=0,$$

therefore $$-12+6\lambda=0, \text{ or } \lambda=2.$$

Hence the complete value of Q is

$$\begin{aligned}\Sigma ab^2.bc^2.ca^2\{(da^4+db^4+dc^4)-(da^2+db^2+dc^2)(ab^2+bc^2+ca^2)\\ +ab^2.bc^2+bc^2.ca^2+ca^2.ab^2\},\\ +2\Sigma(ab^2.bc^2.cd^2.da^2.ac^2);\end{aligned}$$

or, which is the same quantity somewhat differently and more simply arranged,

$$\begin{aligned}Q=\Sigma(ab^2.bc^2.ca^2)\{(da^4+db^4+dc^4+da^2.db^2+db^2.dc^2+dc^2.da)\\ +(ab^2.bc^2+bc^2.ca^2+ca^2.ab^2)-(da^2+db^2+dc^2)(ab^2+bc^2+ca^2)\},\end{aligned}$$

and this quantity equated to zero expresses the conditions of a radius of an inscribed sphere becoming infinite. The direct method would have involved, as the first step, the formation of the Norm of a numerator consisting of

$$\sqrt{F}\pm\sqrt{G}\pm\sqrt{H}\pm\sqrt{K},$$

the value of which is

$$\Sigma F^4-4\Sigma F^3G+6\Sigma F^2G^2+4\Sigma F^2GH-40FGHK,$$

and contains $4+6+12$, *i.e.* 22 positive terms, and 12, *i.e.* 13 negative terms, together 35 terms, each of which might be an aggregate of 6^4 or 1296 quantities, and thus involve in all the consideration of 45360 separate parts, for each of the quantities F, G, H, K being a quadratic function of three of the squared edges, will contain 6 terms. It is not uninteresting to notice that in addition to the case already mentioned of two opposite edges being each zero, as $ab=0$, $cd=0$, Q will also vanish for the case of $ab=cd$, $bc=ad$; *i.e.* for the case of 2 intersecting edges being each equal in length to the edges respectively opposite to them. This is evident from the fact that on the hypothesis supposed the face $acb=acd$ and the face $bdc=bda$; hence $N=0$, and therefore, V not vanishing, $\frac{N}{V^2}$; *i.e.* Q will vanish.

We may moreover remark that since $ab = 0$ and $cd = 0$ does not make V vanish, the perpendicular distance of ab from cd, which, multiplied by $ab \times cd$, gives 6 times the volumes, must on this supposition become infinite. When three edges lying in the same plane all vanish simultaneously, Q vanishes, since one edge at least in every face of the pyramid vanishes, and V also vanishes, as is evident from the expression for V^2, when $ab = 0$, $ac = 0$, $bc = 0$, becoming a multiple of

$$\begin{matrix} 0 & 0 & 0 & ad^2 & 1 \\ 0 & 0 & 0 & bd^2 & 1 \\ 0 & 0 & 0 & cd^2 & 1 \\ ad^2 & bd^2 & cd^2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{matrix}$$

which is evidently zero.

It appeared to me not unlikely, from the situation and look of Q (the characteristic of one of the inscribed spheres becoming infinite), that it might admit of being represented as a determinant, but I have not succeeded in throwing it under that form. I have a strong suspicion that if we take Q' a function corresponding to a tetrahedron $a'b'c'd'$, in the same way as Q corresponds to $abcd$, QQ', and not improbably $\sqrt{(QQ')}$, will be found to be (like as we know from Staud's Theorem of $\sqrt{(V^2.V'^2)}$) a rational integral function of the squares of the distances of the points a, b, c, d, from the points a', b', c', d'.

That N should divide out by V^2 is in itself an analytical theorem relating to 6 arbitrary quantities ab^2, ac^2, ad^2, bc^2, bd^2, cd^2, which evidently admits of extension to any triangular number 10, 15, &c. of arbitrary quantities. Thus we may affirm, *à priori*, that the norm of

$$\sqrt{L} \pm \sqrt{M} \pm \sqrt{N} \pm \sqrt{P} \pm \sqrt{Q},$$

where (for the sake of symmetry, retaining double letters, as AB, AC, &c., to denote *simple* quantities)

$$Q = \begin{vmatrix} 0 & AB & AC & AD & 1 \\ AB & 0 & BC & BD & 1 \\ AC & BC & 0 & CD & 1 \\ AD & BD & CD & 0 & 1 \\ 1 & 1 & 1 & 1 & \end{vmatrix} \quad P = \begin{vmatrix} 0 & AB & AC & AE & 1 \\ AB & 0 & BC & BE & 1 \\ AC & BC & 0 & CE & 1 \\ AE & BE & CE & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

$$N = \&c., \quad M = \&c., \quad L = \&c.,$$

will contain as a factor the determinant

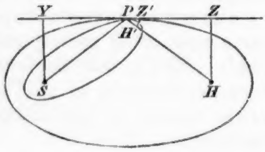
$$\begin{array}{cccccc} 0 & AB & AC & AD & AE & 1 \\ AB & 0 & BC & BD & BE & 1 \\ AC & BC & 0 & CD & CE & 1 \\ AD & BD & CD & 0 & DE & 1 \\ AE & BE & CE & DE & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

and a similar theorem may evidently be extended to the case of any $\frac{n.(n+1)}{2}$ arbitrary quantities whatever.

7, *New Square, Lincoln's Inn,*
March 29, 1853.

SOLUTIONS OF PROBLEMS.

1. Let S be the common focus of the two ellipses, H the second focus of the fixed, H' of the variable one, P their point of contact. Then, P, H', H will be in the same straight line. Let a, b be the semi-axes of the fixed ellipse, a', b' of the variable one. Then, in the first case, a' is constant, and



$$SP + PH = 2a,$$

$$SP + PH' = 2a';$$

therefore $$HH' = 2(a \sim a'),$$

a constant quantity. Hence, the locus of H' is a circle of which H is the centre.

In the second case, b' is constant. Let YZ be the common tangent to the two ellipses at P, and draw SY, HZ, $H'Z'$ perpendicular to YZ. Then

$$SY.HZ = b^2,$$

$$SY.H'Z' = b'^2;$$

therefore $$H'Z' : HZ :: b'^2 : b^2 :: H'P : HP;$$

therefore $$HH' : HP :: b^2 \sim b'^2 : b^2$$

a constant ratio. Hence, the locus of H' is an ellipse similar

and similarly situated to the given one, and having H for a focus.

2. Let a be the radius of the generating circle; the equation to the cycloid, meaning the arc from the vertex, will be

$$s^2 = 8ax.$$

Hence, it may easily be seen* that the equation to the tangent at any point is

$$s = mx + \frac{2a}{m},$$

m denoting the secant of its inclination to the axes of x.

This may be written

$$x \,.\, m^2 - s \,.\, m + 2a = 0:$$

this being a quadratic in m, shews that two tangents may be drawn to the cycloid through any point. If m_1, m_2 be the roots of the above equation, the condition of perpendicularity will be

$$\frac{1}{m_1^{\,2}} + \frac{1}{m_2^{\,2}} = 1;$$

whence, by the theory of equations,

$$\left(\frac{s}{2a}\right)^2 - 2\,\frac{x}{2a} = 1;$$

or

$$s^2 = 4a(x + a),$$

the equation to the required locus, which is evidently a cycloid of half the dimensions of, and similarly placed to the original one.

3. We know that

$$\int_0^\infty \frac{\varepsilon^{av} - \varepsilon^{-av}}{\varepsilon^{\pi v} - \varepsilon^{-\pi v}}\, dv = \tfrac{1}{2}\tan\tfrac{1}{2}\alpha \ldots\ldots\ldots\ldots\ldots (1),$$

and that

$$\int_0^\infty \frac{\varepsilon^{av} - \varepsilon^{-av}}{\varepsilon^{\pi v} - \varepsilon^{-\pi v}}\cos rv dv = \frac{\sin\alpha}{\varepsilon^{r} + 2\cos\alpha + \varepsilon^{-r}} \ldots\ldots (2).$$

Hence, expanding $\cos rv$ in (2) and equating coefficients of $r^{2\mu}$, we have, by Sir John Herschel's theorem,

$$\int_0^\infty \frac{\varepsilon^{av} - \varepsilon^{-av}}{\varepsilon^{\pi v} - \varepsilon^{-\pi v}}\, v^{2\mu}\, dv = \frac{(-1)^\mu\,(1+\Delta)\sin\alpha}{(1+\Delta)^2 + 2(1+\Delta)\cos\alpha + 1}\, o^{2\mu} \ldots (3),$$

* This will follow in precisely the same way as it is shewn that the equation to the tangent to the parabola $y^2 = kax$ is $y = mx + \frac{a}{m}$, where m denotes the tangent of its inclination to the axis of x.

Hence, writing $2x$ for α in (1), and differentiating 2μ times with respect to x, we have by (3)

$$\frac{d^{2\mu}}{dx^{2\mu}}\tan x = \frac{(-1)^{\mu}\, 2^{2\mu+2}\,(1+\Delta)\tan x}{b(1+\Delta)+\Delta^2(1+\tan^2 x)}\, 0^{2\mu}.$$

And in like manner, since

$$\int_0^{\infty}\frac{\varepsilon^{\alpha v}+\varepsilon^{-\alpha v}}{\varepsilon^{\pi v}-\varepsilon^{-\pi v}}\sin rv\,dv = \frac{1}{2}\,\frac{\varepsilon^{r}-\varepsilon^{-r}}{\varepsilon^{r}+2\cos\alpha+\varepsilon^{-r}},$$

we have, equating coefficients of r as before,

$$\int_0^{\infty}\frac{\varepsilon^{\alpha v}+\varepsilon^{-\alpha v}}{\varepsilon^{\pi v}-\varepsilon^{-\pi v}}\, v^{2\mu+1}\,dv = \frac{(-1)^{\mu}}{2}\,\frac{\Delta(\Delta+2)}{(1+\Delta)^2+2(1+\Delta)\cos x+1}\, 0^{2\mu+1}.$$

Hence, we have

$$\frac{d^{2\mu+1}}{dx^{2\mu+1}}\tan x = \frac{(-1)^{\mu}\, 2^{2\mu+1}\,\Delta(\Delta+2)(1+\tan^2 x)}{b(1+\Delta)+\Delta^2(1+\tan^2 x)}\, 0^{2\mu+1}.$$

4. Let ι be the inclination of the plane to the horizon, then the resolved part of the accelerating force of gravity parallel to the plane will be $g\sin\iota = f$ suppose. Let a be the radius of the sphere, M its mass, ω the angular velocity of the plane. Take the point where the axis of revolution meets the plane as origin of coordinates, let the plane itself be that of xy, and the axis of x be parallel to the horizon, x, y the coordinates of the point of the sphere in contact with the plane, F_x, F_y, the resolved part of the friction parallel respectively to the axes of x and y. We have then

$$M\frac{d^2x}{dt^2} = F_x \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

$$M\frac{d^2y}{dt^2} = F_y - Mf \;\ldots\ldots\ldots\ldots\ldots\; (2);$$

and if ω_x, ω_y be the angular velocities of the sphere about axes through its centre parallel respectively to the axes of x and y,

$$Mk^2\frac{d\omega_x}{dt} = F_y a \;\ldots\ldots\ldots\ldots\ldots\ldots\; (3),$$

$$Mk^2\frac{d\omega_y}{dt} = -F_x a \;\ldots\ldots\ldots\ldots\ldots\ldots\; (4).$$

As a further condition, the velocity of the point of the sphere in contact with the plane must be the same as that

of the point of the plane in contact with it. This gives us

$$\frac{dx}{dt} - \omega_y a = -\omega y \text{ .... } (5),$$

$$\frac{dy}{dt} + \omega_x a = \omega x \text{ .... } (6).$$

Differentiating (5), (6), and eliminating ω_x, ω_y, F_x, F_y by (1), (2), (3), (4), we get

$$\frac{d^2x}{dt^2} = -\frac{k^2}{k^2+a^2}\,\omega\,\frac{dy}{dt} \text{ .... } (7),$$

$$\frac{d^2y}{dt^2} = \frac{k^2}{k^2+a^2}\,\omega\,\frac{dx}{dt} - \frac{a^2}{k^2+a^2}f \text{ ...... } (8);$$

whence $$\frac{dx}{dt} = -\frac{k^2}{k^2+a^2}\,\omega y + C \text{ .... } (9);$$

therefore $$\frac{d^2y}{dt^2} = -\frac{k^4}{(k^2+a^2)^2}\,\omega^2 y + \frac{k^2}{k^2+a^2}\,\omega C - \frac{a^2}{k^2+a^2}f \text{...} (10).$$

To determine C, let x_0, y_0 be the initial values of x, y; and let I_x, I_y be the impulses of the friction when the sphere is just placed in the plane; $\left(\frac{dx}{dt}\right)_0$, $\left(\frac{dy}{dt}\right)_0$, $(\omega_x)_0$, $(\omega_y)_0$ the corresponding values of $\frac{dx}{dt}$, &c., then

$$M\left(\frac{dx}{dt}\right)_0 = I_x \text{ .... } (1'),$$

$$M\left(\frac{dy}{dt}\right)_0 = I_y \text{ .... } (2'),$$

$$Mk^2(\omega_x)_0 = I_y a \text{ .... } (3'),$$

$$Mk^2(\omega_y)_0 = -I_x a \text{ .... } (4'),$$

and, as in (5), (6),

$$\left(\frac{dx}{dt}\right)_0 - (\omega_y)_0 a = -\omega y_0 \text{ .... } (5'),$$

$$\left(\frac{dy}{dt}\right)_0 + (\omega_x)_0 a = \omega x_0 \text{ .... } (6'),$$

therefore, by (1'), (4'), (5'),

$$\left(1+\frac{a^2}{k^2}\right)\left(\frac{dx}{dt}\right)_0 = -\omega y_0,$$

Similarly, by (2′), (3′), (6′),

$$\left(1+\frac{a^2}{k^2}\right)\left(\frac{dy}{dt}\right)_0 = \omega x_0.$$

Hence (9) gives $C = 0$,

$$\therefore \text{ by (10)}, \quad \frac{d^2y}{dt^2} = -\frac{k^2}{(k^2+a^2)^2}\omega^2 y - \frac{a^2}{k^2+a^2}f;$$

whence $y = -\dfrac{(k^2+a^2)a^2}{k^4}\dfrac{f}{\omega^2} + A\cos\left(\dfrac{k^2}{k^2+a^2}\omega t + B\right)$,

$$\therefore \left(\frac{dy}{dt}\right) = -\frac{k^2}{k^2+a^2}\omega A \sin\left(\frac{k^2}{k^2+a^2}\omega t + B\right).$$

To determine the constants, we have

when $t = 0$, $y = y_0$, $\dfrac{dy}{dt} = \dfrac{k^2}{k^2+a^2}\omega x_0$,

$$\therefore y_0 = -\frac{(k^2+a^2)a^2}{k^2}\frac{f}{\omega^2} + A\cos B,$$

$$\frac{k^2}{k^2+a^2}\omega x_0 = -\frac{k^2}{k^2+a^2}\omega A\sin B;$$

$$\therefore y = -\frac{(k^2+a^2)a^2}{k^4}\frac{f}{\omega^2} + \left\{y_0 + \frac{(k^2+a^2)a^2}{k^4}\frac{f}{\omega^2}\right\}$$
$$\cos\frac{k^2}{k^2+a^2}\omega t + x_0\sin\frac{k^2}{k^2+a^2}\omega t \;\ldots (11);$$

and, by (9),

$$\frac{dx}{dt} = \frac{a^2}{k^2}\frac{f}{\omega} - \left(\frac{k^2}{k^2+a^2}\omega y_0 + \frac{a^2}{k^2}\frac{f}{\omega}\right)$$
$$\cos\frac{k^2}{k^2+a^2}\omega t - \frac{k^2}{k^2+a^2}\omega x_0 \sin\frac{k^2}{k^2+a^2}\omega t;$$

therefore, integrating,

$$x = \frac{a^2}{k^2}\frac{f}{\omega}t + x_0\cos\frac{k^2}{k^2+a^2}\omega t - \left\{y_0 + \frac{(k^2+a^2)a^2}{k^4}\frac{f}{\omega^2}\right\}\sin\frac{k^2}{k^2+a^2}\omega t$$
$$\ldots\ldots\ldots(12).$$

The equations (11), (12) give the path of the centre of the sphere in space. It will be a trochoid whose axis is parallel to that of x.

6. Let the density of the matter at C, the centre of the sphere, be denoted by k, and let f be the distance of this

point from the given external point S. The expression for the density at any point Π, of which the distance from f is D, will be

$$\frac{kf^5}{D^5} \text{ ....} (1).$$

If we take polar coordinates, ρ, ϑ, ϕ for the point Π, the corresponding expression for the volume of an element of the solid will be $\rho^2 \sin\vartheta \, d\phi \, d\vartheta \, d\rho$, and the quantity of matter which it contains will be

$$\frac{kf^5}{D^5} \cdot \rho^2 \sin\vartheta \, d\phi \, d\vartheta \, d\rho \text{ ....} (2).$$

The potential due to this at any point P, at a distance Δ from it, will be

$$\frac{kf^5 \, \rho^2 \sin\vartheta \, d\phi \, d\vartheta \, d\rho}{D^5 . \Delta} \text{ ....} (3),$$

and if r, θ, φ be the coordinates of P, we shall have

$$\Delta = [\rho^2 - 2\rho r \{\cos\vartheta \cos\theta + \sin\vartheta \sin\theta \cos(\phi - \varphi)\} + r^2]^{\frac{1}{2}} \ldots (4).$$

A similar expression might be used for D, but if we take S for the origin of polar coordinates, we have simply

$$D = \rho \text{ ....} (5).$$

If, farther, we take SC as the polar axis, we shall have for the equation of the surface of the sphere,

$$\rho^2 - 2f\rho \cos\vartheta + f \;\; = a^2 \text{ ....} (6),$$

where a denotes the radius.

If now we denote by V the potential at P due to the entire sphere, we deduce from the preceding expressions

$$V = kf^5 \iiint \frac{\sin\vartheta \, d\phi \, d\vartheta \, d\rho}{\rho^3 [\rho^2 - 2\rho r \{\cos\vartheta \cos\theta + \sin\vartheta \sin\theta \cos(\phi - \varphi)\} + r^2]} \ldots (7),$$

where the integration is to be made so as to include all values of ρ, ϑ, and ϕ subject to the condition

$$\rho^2 - 2f\rho \cos\vartheta + f^2 < a^2 \text{ ....} (8).$$

The evaluation of this integral may be effected by means of the following simple transformation:

Let $$\rho = \frac{f^2 - a^2}{\rho'} \text{ ....} (9);$$

then, $$\frac{d\rho}{\rho^3} = - \frac{\rho' d\rho'}{(f^2 - a^2)^2} \text{ ....} (10),$$

if, besides, we assume $r = \dfrac{f^2 - a^2}{r'}$ ....(11),

we shall have

$$\Delta = \frac{f^2 - a^2}{\rho' r'}\,[\rho'^2 - 2\rho' r'\{\cos\vartheta\cos\theta + \sin\vartheta\sin\theta\cos(\phi - \varphi)\} + r'^2]^{\frac{1}{2}} \ldots (12).$$

The expression for V then becomes

$$V = \frac{kf^5 r'}{(f^2 - a^2)^3}\iiint \frac{\rho'^2 \sin\vartheta\, d\phi\, d\vartheta\, d\rho'}{[\rho'^2 - 2\rho' r'\{\cos\vartheta\cos\theta + \sin\vartheta\sin\theta\cos(\phi - \varphi)\} + r'^2]^{\frac{1}{2}}} \ldots (13),$$

with the limiting condition

$$\frac{(f^2 - a^2)^2}{\rho'^2} - 2f\frac{f^2 - a^2}{\rho'}\cos\vartheta + f^2 < a^2 \ldots\ldots (14),$$

which is equivalent to

$$(f^2 - a^2)^2 - 2f\rho'\cos\vartheta\,(f^2 - a^2) + \rho'^2(f^2 - a^2) < 0\,;$$

or to $\qquad f^2 - a^2 - 2f\rho'\cos\vartheta + \rho'^2 < 0\,;$

or, lastly, $\qquad \rho'^2 - 2f\rho'\cos\vartheta + f^2 < a^2$...............(15).

Now the triple integral in the second members of (13) is, as we see immediately, the expression for the potential at a point (r', θ, φ), of a mass of the uniform density unity, occupying all the space over which the integration with reference to ρ', ϑ, ϕ, considered as polar coordinates, is extended. The equation of the surface bounding this space is, according to (14),

$$\rho^2 - 2f\rho\cos\vartheta + f^2 = a^2,$$

which agrees with (6), that of the given sphere. Hence, assuming the known results with reference to the attraction of a uniform sphere, we may immediately obtain the value of the triple integral in (13). Thus, according as the point (r', θ, φ) is without or within the surface of the sphere, we shall have for the expression,

$$V = \frac{kf^5 r'}{(f^2 - a^2)^3}\cdot\frac{\frac{4\pi}{3}a^3}{(r'^2 - 2fr'\cos\theta + f^2)^{\frac{1}{2}}}\,;$$

when $\qquad r'^2 - 2fr'\cos\theta + f^2 > a^2\,;$

or $\qquad V = \dfrac{kf^5 r'}{(f^2 - a^2)^3}\cdot\dfrac{4\pi}{3}(r'^2 - 2fr'\cos\theta + f^2)\,;$

when $\qquad r'^2 - 2fr'\cos\theta + f^2 < a^2.$

If now we recur to the original notation, by substituting $\frac{f^2 - a^2}{r}$ for r' in these expressions, and modify the discriminating conditions as we have done above, in the case of (14), we obtain

$$(A) \ldots V = \frac{\frac{4\pi}{3} a^3 k f^4}{(f^2 - a^2)^2} \cdot \frac{1}{\left\{\left(\frac{f^2 - a^2}{f^4}\right)^2 - 2\left(\frac{f^2 - a^2}{f}\right) r \cos\theta + r^2\right\}^{\frac{1}{2}}};$$

when $$r^2 - 2fr \cos\theta + f^2 > a^2,$$

$$(B) \ldots V = \frac{\frac{4\pi}{3} k f^7}{(f^2 - a^2)^2} \cdot \frac{\left(\frac{f^2 - a^2}{f}\right)^2 - 2\left(\frac{f^2 - a^2}{f}\right) r \cos\theta + r^2}{r^3};$$

when $$r^2 - 2fr \cos\theta + f^2 < a^2.$$

These expressions, for the potential without and within the sphere, contain the complete solution of the problem, since the two components, (F) along PS, and (G) perpendicular to PS in the plane PST, of the force at any point P, may be found from them by means of the equations

$$(C) \ldots F = \frac{dv}{dr}, \quad \text{and } G = \frac{1}{r}\frac{dv}{d\theta}.$$

If we put, for brevity,

$$\frac{\frac{4\pi}{3} a^3 k f^4}{(f^2 - a^2)} = m \text{ ....} (16),$$

and $$\frac{f^2 - a^2}{f} = g \text{ ....} (17),$$

the expressions for the potential become simply,

(external point), $$V = \frac{m}{(g^2 - 2gr \cos\theta + r^2)^{\frac{1}{2}}} \text{ ....} (18),$$

(internal point), $$V = \frac{mf^3}{a^3} \cdot \frac{g^2 - 2gr \cos\theta + r^2}{r^3} \text{ ....} (19).$$

The denominator of the former expression being obviously the distance from P to a certain point I, taken in SC at a distance $g = \frac{f^2 - a^2}{f}$ from S, we conclude that the resultant force is towards this point I, and inversely as the square of the distance of P from it.

7. When a gas expands without being allowed to take in any heat from without, its pressure varies as the m^{th} power of its density, provided only m is constant. Hence, if v be the volume, and p the pressure of the expanding air at any instant during the motion of the ball through the barrel, we have

$$p = P\left(\frac{V}{v}\right)^m.$$

Now the expanding air increases in volume during the motion of the ball from one end of the barrel to the other, by U, the volume of the barrel, and therefore the work done by it is $\int_V^{U+V} n\,dv$, or, as we find by integration from the preceding equation,

$$\frac{PV}{m-1}\left\{1 - \left(\frac{V}{U+V}\right)^{m-1}\right\}.$$

According to the hypotheses, the whole of this work is spent in pushing out the ball, and produces only the two effects of overcoming the atmospheric pressure in front of the ball, and communicating motion to the ball. The mechanical value of the first of these effects is ΠU, and that of the second $\frac{1}{2}\frac{W}{g}q^2$, if q be the velocity acquired by the ball. Hence, by the principle of *vis viva*,

$$\tfrac{1}{2}\frac{W}{g}q^2 + \Pi U = \frac{PV}{m-1}\left\{1 - \left(\frac{V}{U+V}\right)^{m-1}\right\},$$

from which we derive

$$q = \left[\frac{2g}{W}\overline{\frac{PV}{m-1}\left\{1 - \left(\frac{V}{U+V}\right)^{m-1}\right\} - \Pi U}\right]^{\frac{1}{2}},$$

the required expression, (as corrected in "Errata" published in the present Number).

Let $U + u$ be the volume the barrel would require to have that the pressure of the air in it may be just equal to Π, the atmospheric pressure at the instant when the ball leaves it. Then,

$$\left(\frac{V+U+u}{V}\right)^m = \frac{P}{\Pi}, \quad \text{whence } u = V\left\{\left(\frac{P}{\Pi}\right)^{\frac{1}{m}} - \frac{U+V}{V}\right\};$$

or, if Q be the pressure in the actual barrel, (of which the

volume is U) at the instant when the ball leaves it,

$$u = V\left\{\left(\frac{P}{\Pi}\right)^{\frac{1}{m}} - \left(\frac{P}{Q}\right)^{\frac{1}{m}}\right\},$$

a quantity which will in general be positive since in all good practical arrangements Q must exceed Π. Now, if the volume of the barrel were $U + u$, there would be no appreciable proportion of the whole work spent in noise &c., since the mechanical value of the motion of the air in the barrel till the ball leaves it, is, according to the hypotheses, to be neglected in every case, and the only energy that remains to produce noise &c., would in this case be the motion of the air within the barrel. But in this case the whole work spent in communicating motion to the ball would be

$$\frac{PV}{m-1}\left\{1 - \left(\frac{V}{U+V+u}\right)^{m-1}\right\} - \Pi\,(U+u),$$

which exceeds the work spent in communicating motion to the ball in the actual case by

$$\frac{PV}{m-1}\left\{\left(\frac{V}{U+V}\right)^{m-1} - \left(\frac{V}{U+V+u}\right)^{m-1}\right\} - \Pi u,$$

or $$\frac{PV}{m-1}\cdot\frac{Q^{\frac{m-1}{m}} - \Pi^{\frac{m-1}{m}}}{P^{\frac{m-1}{m}}} - \Pi V\left\{\left(\frac{P}{\Pi}\right)^{\frac{1}{m}} - \left(\frac{P}{Q}\right)^{\frac{1}{m}}\right\}.$$

Now the final effect, in lifting the atmosphere in the two cases is the same, (with the exception of small differences that may result from differences of temperature of the air near the mouth of the gun, which are neglected), and hence, the excess of work spent in communicating motion to the ball, in one case is equal to that wasted in noise and fluid friction after the ball leaves the gun in the other; and therefore the ratio required in the second part of the question is

$$\frac{\dfrac{PV}{m-1}\,\dfrac{Q^{\frac{m-1}{m}} - \Pi^{\frac{m-1}{m}}}{P^{\frac{m-1}{m}}} - \Pi V\left\{\left(\dfrac{P}{\Pi}\right)^{\frac{1}{m}} - \left(\dfrac{P}{Q}\right)^{\frac{1}{m}}\right\}}{\dfrac{PV}{m-1}\left\{1 - \left(\dfrac{Q}{P}\right)^{\frac{m-1}{m}}\right\} - \Pi V\left(\dfrac{P}{Q}\right)^{\frac{1}{m}}}.$$

NOTE—Solutions of problems (5), (8), (9), (10) will appear in the next number.

PROBLEMS.

1. Given two conics in the same plane such that the normal distance of the point of intersection of their transverse or major axes from each of the conics is one and the same pure imaginary quantity; shew that the conics may be projected into small circles of the same sphere.

2. If from a point in the circumference of a vertical circle two heavy particles be successively projected along the curve, then initial velocities being equal and either in the same or in opposite directions, the subsequent motion will be such that a straight line joining the particles at any instant will touch a given circle.

Note. The particles are supposed not to interfere with each other's motion.

3. A transparent medium is such that the path of a ray of light within it is a given circle, the index of refraction being a function of the distance from a given point in the plane of the circle.

Find the form of this function and shew that for light of the same refrangibility—

(1) The path of *every ray within the medium* is a circle.

(2) All the rays proceeding from any point in the medium will meet accurately in another point.

(3) If rays diverge from a point without the medium and enter it through a spherical surface having that point for its centre, they will be made to converge accurately to a point within the medium.

4. A series of waves, which at sea are twenty feet long from crest to crest, and three feet high from hollow to crest, break on a shore which is parallel to their breadth. How much heat is developed per hour on each foot of the shore, and how much would the temperature of 180 cubic feet of fresh water be raised by receiving an equal quantity? [The form of a wave at sea, of which the height is a small fraction of l, its length, is approximately the curve of sines; its velocity of propagation is $\sqrt{\frac{gl}{2\pi}}$; and its mechanical energy is half that of a double elevation and depression of the same form without velocity.]

5. (*a*) What horse-power would be required to supply a building with 1 lb. of air per second, heated mechanically from 50° to 80° Fahrenheit? Compare the fuel that an engine producing this effect as $\frac{1}{10}$ of the equivalent of the heat of combustion would consume, with that which would be required to heat directly the same quantity of air.

(*b*) Explain how this effect may be produced with perfect economy by operating on the air itself to change its temperature, and give dimensions &c. of an apparatus that may be convenient for the purpose.

(*c*) Shew how the same apparatus may be adapted to give a supply of cooled air.

Ex. Let it be required to supply a building with 1 lb. of air per second, cooled from 80° to 50° Fahr. Determine the horse-power wanted to work the apparatus in this case.

6. Find the amount of "potential energy" (mechanical effect of such a kind as that of weights raised) that can be obtained by equalizing the temperature of two bodies given at different uniform temperatures, and determine the common temperature to which they are reduced.

Ex. 1. Let the bodies be of equal constant thermal capacities, and let their temperatures be 0° and 100° respectively.

Ex. 2. Let the bodies be masses W and W' of water, and let the temperatures at which they are given be 15° and 20° respectively.

7. If α, β, γ, be the trilinear coordinates of a point, a, b, c, the lengths of the sides of the triangle of reference, the equations to the greatest inscribed and least circumscribed ellipse will be respectively

$$(a\alpha)^{\frac{1}{2}} + (b\beta)^{\frac{1}{2}} + (c\gamma)^{\frac{1}{2}} = 0,$$

$$(a\alpha)^{-1} + (b\beta)^{-1} + (c\gamma)^{-1} = 0.$$

ON THE THIRD ELLIPTIC INTEGRAL.

By F. W. Newman, formerly Fellow of Balliol College, Oxford.

I. Following up the splendid discoveries of Jacobi, Legendre first, and since his labours were closed, Dr. Gudermann, have investigated series of an elevated kind for approximating to this integral. But the higher theory seems to have drawn off investigations unduly from what is more elementary; and the principal object of the present paper is to shew that the earlier and simpler methods have by no means been adequately appreciated and developed.

Legendre's notation, with trifling alterations, will be here retained. The moduli $cc_1c_2c_3\ldots$ are those of the common scale *descending*, which he denotes by $cc^{\circ}c^{\circ\circ}c^{\circ\circ\circ}\ldots$ I propose to employ the notation $\eta\eta^{\circ}\eta_{\circ}$ to imply the relations

$$F(c\eta) + F(c\eta^{\circ}) = F(c, \tfrac{1}{2}\pi),$$
$$F(b\eta) + F(b\eta_{\circ}) = F(b, \tfrac{1}{2}\pi),$$

in which case η° may be called the *conjugate* amplitude to η, and $\eta_{\circ}$ the *lower conjugate*. We then have the well-known relations $\cot\eta.\cot\eta^{\circ} = b$,

$$\sin\eta^{\circ} = \frac{\cos\eta}{\Delta(c\eta)}, \quad \cos\eta^{\circ} = \frac{b\sin\eta}{\Delta(c\eta)}, \quad \Delta(c\eta^{\circ}) = \frac{b}{\Delta(c\eta)} \ldots (1),$$

in which we may change η°, c into $\eta_{\circ}, b$.

Also if from c, η be formed c_1, η_1 in Lagrange's scale, we get

$$\left.\begin{array}{ll} \sqrt{c_1}\sin\eta_1 = \sqrt{c}\sin\eta.\sqrt{c}\sin\eta^{\circ}; & \Delta(c\eta)+\Delta(c\eta^{\circ}) = (1+b)\Delta(c_1\eta_1) \\ & \Delta(c\eta)-\Delta(c\eta^{\circ}) = (1-b)\cos\eta_1 \\ \cos\eta_1 = \sin(\eta^{\circ}-\eta),\ldots & \text{or } \eta_1 = \tfrac{1}{2}\pi + \eta - \eta^{\circ} \end{array}\right\}\ldots(2).$$

For the complete integrals $F(c, \frac{1}{2}\pi)$, $E(c, \frac{1}{2}\pi)$, &c.... we may generally conform to a prevalent method of writing them F_c, E_c, &c. But in the case of F_c it is sometimes necessary to break the analogy (as in the higher theory) by writing C: for when c changes to $c\,c_1c_2\ldots b\,b_1b_2\ldots$, to write F_{c_1}, F_{c_2}, &c. is very incommodious.

II. Legendre's results concerning the integral Π_1 may be summed up nearly as follows:

(1) That every Π which has a parameter of the form $\alpha + \beta\sqrt{-1}$ is reducible to *two* Π's with real parameters, and with coefficients of the form $\alpha + \beta\sqrt{-1}$.

(2) If $Q = \dfrac{\tan\omega}{\Delta(c\omega)} = \dfrac{\cos\omega^{\circ}}{b\cos\omega}$, and $T = (1+p)\left(1 + \dfrac{c^2}{p}\right)$,

$$\therefore \int \frac{dQ}{1+TQ^2} = \Pi(p) + \Pi\left(\frac{c^2}{p}\right) - F \text{ .... } (3),$$

where p, c^2p^{-1} are parameters, and c, ω the other elements.

Let $pq = c^2$, then p and q are called *reciprocal* in this theory.

(3) If $(1+p)(1-r) = b^2$, therefore

$$\int_0 \frac{d(\sin\omega \sin\omega^\circ)}{1+pr(\sin\omega\sin\omega^\circ)^2} = \frac{1+p}{p}.\Pi(p) - \frac{1-r}{r}.\Pi(-r) - \frac{c^2}{pr}.F \ldots (4).$$

The parameters p and $-r$ are called *conjugate*.

(4) Various integrals $P(p,\omega) = \int \phi(p,\omega)\,d\omega$ are known, which may be found in some simpler form for *a special* value of ω, $(\omega = \alpha)$ by means of

$$\frac{dP}{dp} = \int \frac{d\phi(p,\omega)}{dp}\,d\omega = \psi(p,\omega).$$

If the function ψ is known, we get

$$\frac{dP(p,\alpha)}{dp} = \psi(p,\alpha), \text{ and } P(p,\alpha) = \int \psi(p,\alpha)\,dp.$$

Legendre applied this method to Π, and deduced not only the value of Π_c in terms of F and E, but certain *commutative* equations, in which the amplitude exchanges places with a certain function of the parameter.

(5) He applied Lagrange's scale to Π, and by it deduced two series, one for descending, the other for ascending, moduli. But both are too complicated for use, especially the latter.

One more property established by Legendre remains to be named; viz. if ζ, ω, η are *amplitudes* which make $F\zeta = F\omega + F\eta$, and p the parameter common to the three Π's which correspond to the F's, then

$$\sqrt{T}.\{\Pi\omega + \Pi\eta - \Pi\zeta\} = \tan^{-1}.\frac{\sqrt{T}.p\sin\omega\sin\eta\sin\zeta}{1+p(1-\cos\omega\cos\eta\cos\zeta)} \text{ ......} (5).$$

Consequently if $\zeta = \frac{1}{2}\pi$, or $\eta = \omega^\circ$, we get

$$\sqrt{T}\{\Pi\omega + \Pi\omega^\circ - \Pi_c\} = \tan^{-1}.\left\{\frac{p}{1+p}.\sqrt{T}.\sin\omega\sin\omega^\circ\right\} \ldots (6),$$

in which, whenever T is negative, it will be easy to give to the last term the form of a logarithm.

My first business is, to shew that all these integrations of Legendre, when duly simplified, lead to available results.

III. The function $T=(1+p)(1+q)$ may be called the *test product*, since, according as it is negative or positive, Π is of the logarithmic or of the circular class. We may occasionally denote it by $T(p)$, and the definition shews that $T(p)=T(q)$: also

$$T(p).T(-r)=b^4 \text{ .... } (7).$$

It is easy to prove that the reciprocals of conjugate parameters are conjugate, and the conjugates of reciprocals are reciprocal. Thus the reciprocal of the conjugate is the conjugate of the reciprocal.

Also two reciprocals, or two conjugates, are either both circular or both logarithmic.

Legendre assigns two forms for logarithmic parameters, viz. $-c^2\sin^2\eta$ and $-\mathrm{cosec}^2\eta$, both negative; the former ranging from 0 to $-c^2$, the latter from -1 to $-\infty$. Evidently if a parameter has the form $-c^2\sin^2\eta$, its reciprocal is $-\mathrm{cosec}^2\eta$. But logarithmic *conjugates* are either both of the form $-c^2\sin^2\eta$ or both of the form $-\mathrm{cosec}^2\eta$. In fact, since $\Delta(c\eta).\Delta(c\eta^\circ)=b$, it follows that $-c^2\sin^2\eta$ and $-c^2\sin^2\eta^\circ$ are logarithmic conjugates. So also, since $\cot\eta.\cot\eta^\circ=b$, therefore $-\mathrm{cosec}^2\eta$ and $-\mathrm{cosec}^2\eta^\circ$ are logarithmic conjugates.

A circular parameter, when positive, may be denoted by $p=\cot^2\theta$; then its reciprocal is $q=c^2\tan^2\theta$, and its conjugate is $-r=-1+b^2\sin^2\theta$. But we may also write $p=\cot^2\theta$, $q=\cot^2\theta_\circ$, $r=\Delta^2(b,\theta)$. Of two circular conjugates one is necessarily negative, the other positive. Also, since

$$\Delta(b,\theta)\,\Delta(b,\theta_\circ)=c,$$

the two circular parameters $-\Delta^2(b,\theta)$ and $-\Delta^2(b,\theta_\circ)$ are reciprocal.

In the equations (5), (6) every Π is multiplied by $\sqrt{T}$; and in the farther development of the theory the same phenomenon constantly recurs; insomuch that $\sqrt{T}\Pi$ seems (rather than Π) to be the function which we are concerned with. In this connection it is highly interesting to find, that the curves drawn on surfaces of the second order, which are said to be measured by this integral, are really measured by the compound function $\sqrt{T}.\Pi$. (See Dr. James Booth, *Philos. Trans.* 1852, p. 320, equations (17), (18), &c.) Of course, when T is negative, we must deal with $\sqrt{-T}.\Pi$.

When T is separated from Π, it may be requisite, as above, to write the parameter after T; as $T(p)$ for $(1+p)(1+c^2p^{-1})$, &c. But whenever T is *immediately* followed by Π, (or by P, of which I proceed to speak,) it will be understood that T involves the *same* parameter and modulus as the Π, or

as the P. Thus $\sqrt{T}\Pi(p) + \sqrt{T}\Pi(-r)$ will mean

$$\sqrt{T(p)}.\Pi(p) + \sqrt{T(-r)}.\Pi(-r).$$

When p is infinitesimal, $\Pi = F$; also $T = c^2 p^{-1}$; whence

$$\sqrt{T}(F - \Pi) = c\sqrt{p^{-1}} \textstyle\int_0 p \sin^2\omega . dF = 0.$$

Also when p is infinite, $T = p$, and $\sqrt{T}\Pi$ has no increments while ω is finite; for then $\dfrac{\sqrt{p}}{1 + p\sin^2\omega} = 0$. But while ω is infinitesimal $\sqrt{T}\Pi = \displaystyle\int_0 \frac{\sqrt{p}d\omega}{1 + p\omega^2} = \tan^{-1}(\sqrt{p}.\omega)$, which becomes $\tan^{-1}(\infty)$, as soon as ω rises to a sensible value; hence $\sqrt{T}\Pi = \frac{1}{2}\pi$, when $p = \infty$, whatever the finite value of ω. This reasoning is rather refined; but the conclusion may be equally obtained from the reciprocal equation.

IV. The three integrals F, E, Π have in common the property, that whenever their amplitude $\omega = n.\frac{1}{2}\pi$, the integral $= n$ times the complete integral. It immediately follows, that if for a moment we assume three arcs x, x', x'' such that

$$\frac{F(\omega)}{F_c} = \frac{x}{\frac{1}{2}\pi},\quad \frac{E(\omega)}{E_c} = \frac{x'}{\frac{1}{2}\pi},\quad \frac{\Pi(\omega)}{\Pi_c} = \frac{x''}{\frac{1}{2}\pi},$$

each of the three new arcs is equal to ω, as often as ω is a multiple of $\frac{1}{2}\pi$. Hence $(x' - x)$ and $(x'' - x)$, or any functions proportional to them, vanish periodically every time that $\omega = n.\frac{1}{2}\pi$. Such *fluctuating* functions are the appropriate auxiliaries for calculating E and Π, when F is known.

Legendre assumed G as an auxiliary, equivalent to $E - \dfrac{F}{F_c}.E_c$; which is proportional to $x' - x$; and by it he obtained by far the most elegant of the approximations to E; namely, if C now stands for F_c, he found

$$\left.\begin{aligned} CG &= C_1G_1 + C_1c_1\sin\omega_1 \quad \text{[Lagrange's scale]} \\ \text{whence}\quad CG &= C_1c_1\sin\omega_1 + C_2c_2\sin\omega_2 + C_3c_3\sin\omega_3 + \&\text{c}.... \end{aligned}\right\}\ldots(8).$$

But the properties and uses of G have by no means been fully exhibited, and a digression on that subject, either here or afterwards, is inevitable. If we assume H as a second auxiliary, such that

$$H = E - \left(1 - \frac{E_b}{F_b}\right)F \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

we easily get [since by Legendre's equation of complementary moduli, $\frac{1}{2}\pi = F_bE_c + F_cE_b - F_bF_c$,]

therefore $$H = G + \frac{\frac{1}{2}\pi F}{F_b F_c};$$
or, in the other notation,
$$H = G + \tfrac{1}{2}\pi \cdot \frac{F}{BC} \quad\text{.....................(10).}$$
Moreover $$CH = C_1 H_1 + C_1 c_1 \sin\omega_1;$$
or $$BH = \tfrac{1}{2}B_1 H_1 + \tfrac{1}{2}B_1 c_1 \sin\omega_1 \quad\text{............(11),}$$
whence
$$\left.\begin{aligned} BH = B\sin\omega - 2B'(\sin\omega - \sin\omega') - 2^2B''(\sin\omega' - \sin\omega'') \\ - 2^3B'''(\sin\omega'' - \sin\omega''') - \&c. \end{aligned}\right\}\text{...(12),}$$
which is by far the most elegant series for calculating E by *ascending* moduli, (*i.e.* when c is near to 1,) and converges with the usual precipitancy of Lagrange's scale.

Farther, if $\sin\eta = \sqrt{-1}\tan\theta$, we easily obtain

$$\left.\begin{aligned} &G(c\eta) = -\sqrt{-1} \,.\, H(b,\theta) + \sqrt{-1} \,.\, \tan\theta \,.\, \Delta(b,\theta), \\ &H(c\eta) = -\sqrt{-1} \,.\, G(b,\theta) + \sqrt{-1} \,.\, \tan\theta \,.\, \Delta(b,\theta), \\ \text{or}\quad &G(c\eta) - \sqrt{-1} \,.\, G(b,\theta) = H(c\eta) - \sqrt{-1} \,.\, H(b,\theta), \\ \text{also}\quad &G(c\eta) + \sqrt{-1} \,.\, G(b,\theta) = \sqrt{-1}\left\{\tan\theta\,\Delta(b,\theta) - \tfrac{1}{2}\pi \,.\, \frac{F(b\theta)}{BC}\right\} \end{aligned}\right\}\text{...(13).}$$

Finally, it is worth observing, that while $G(c\omega)$ vanishes, not only when ω is a multiple of $\frac{1}{2}\pi$, but also when c is evanescent; we have, on the other hand, $BH_c = \frac{1}{2}\pi$, for all values of c; also, for all values of ω, we find $BH = \omega$ when $c = 0$; but $H = \sin\omega$, when $c = 1$. We may add that $G(c, n\pi + \omega) = G(c, \omega)$, but $BH(c, n\pi + \omega) = BH(c, \omega) + n\pi$, or $H(c, n\pi + \omega) = H(c, \omega) + 2nH_c$.

V. Returning to the integral Π, we follow out the analogy of this proceeding, by assuming an auxiliary proportional to $(x'' - x)$.

Let P stand for $\Pi - \dfrac{F}{F_c}\Pi_c$......................(14),

then the problem of finding Π divides itself into two parts. First, to find the complete integral Π_c: for when this is known, we regard the second term of P to be known. Next, it remains to find the fluctuating portion P, which alone involves three elements; and since it periodically vanishes, we may look on it as a small correction to be applied to the main term; the total value of Π being given by the equation
$$\Pi = \frac{F}{F_c}\Pi_c + P.$$

It is evident that P vanishes with p. But we must first dispose of the case to which this method is essentially inapplicable, viz. that in which Π_c is infinite; namely in which the parameter has the form $p = -\operatorname{cosec}^2\eta$. By the reciprocal equation (3) Legendre reduces this to the indefinite integral $\Pi(-c^2\sin^2\eta)$: nevertheless, it is not amiss to exhibit the equation in a slightly changed form.

When $p = -\operatorname{cosec}^2\eta$, $T = -\cot^2\eta.\Delta^2(c\eta)$, which applies alike to p and q: therefore

$$\sqrt{-T}\{\Pi(p) + \Pi(c^2p^{-1}) - F\}$$

$$= \int_0 \frac{\sqrt{-T}.dQ}{1+TQ^2} = \tfrac{1}{2}\log\frac{\tan\eta\Delta\omega + \tan\omega\Delta\eta}{\pm\{\tan\eta\Delta\omega - \tan\omega\Delta\eta\}}.$$

Consequently, if $F\zeta = F\omega + F\eta$ and $F\varepsilon = F\omega - F\eta$, the relations, furnished by Euler's well-known integration, between $\zeta\varepsilon$ and $\omega\eta$ yield

$$\sqrt{-T}\{\Pi(-\operatorname{cosec}^2\eta) + \Pi(-c^2\sin^2\eta) - F\}(c\omega) = \tfrac{1}{4}\log\frac{\sin^2\zeta}{\sin^2\varepsilon} \;\ldots(15),$$

in which $\Pi(-\operatorname{cosec}^2\eta)$ and the logarithm both become infinite at the crisis $\omega = \eta$, $\varepsilon = 0$. In future, we set aside the case of parameters negative and greater than unity, as sufficiently disposed of by this equation.

Passing to the circular Π, we may doubly modify the reciprocal equation by supposing p positive or negative. But it will suffice to make p positive, and to treat a negative parameter $(-r)$ as its conjugate. Generally, when T is positive, equation (3) may take the form

$$\sqrt{T}.\{\Pi(p) + \Pi(c^2p^{-1}) - F\} = \tan^{-1}.(\sqrt{T}Q).$$

But when $p = \cot^2\theta$,

$$\sqrt{T} = \frac{\Delta(b\theta)}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\sin\theta_\circ} \;\ldots\ldots\ldots\ldots(16),$$

an expression which is very easy to remember: and the corresponding value of $\sqrt{T}(-r)$, the conjugate, is no additional burden to the memory, if we do but remember the relation $\sqrt{T}(p).\sqrt{T}(-r) = b^2$, from equation (7). Hence

$$\sqrt{T}.Q = \frac{\Delta(b\theta)}{\sin\theta\cos\theta}\cdot\frac{\tan\omega}{\Delta(c\omega)} = \frac{1}{\sin\theta\sin\theta_\circ}\cdot\frac{\cos\omega^\circ}{b\cos\omega} \;\ldots(17).$$

Again, when we assume $\omega = \tfrac{1}{2}\pi$, $Q = \infty$, $\tan^{-1}(\sqrt{T}Q) = \tfrac{1}{2}\pi$; whence

$$\sqrt{T}\{\Pi_c(p) + \Pi_c(c^2p^{-1}) - F_c\} = \tfrac{1}{2}\pi \;\ldots\ldots\ldots\ldots(17a).$$

Multiply this by $\dfrac{F\omega}{F_c}$ and subtract the product from the general integral; therefore

$$\left.\begin{aligned}\sqrt{T}\left\{P(p)+P(c^2p^{-1})\right\}(c\omega) = \cot^{-1}\left\{\sin\theta\sin\theta_\circ.\,\frac{b\cos\omega}{\cos\omega^\circ}\right\}\\ -\tfrac{1}{2}\pi.\,\frac{F(c\omega)}{F_c}\end{aligned}\right\}\ldots(18),$$

when $p=\cot^2\theta$.

VI. From the reciprocal we proceed to the conjugate equation (4).

When Π is logarithmic, p as well as $-r$ will be negative, and we may write $-r'$ for p, so that we get

$$\left.\begin{array}{l}R=\sin\omega\sin\omega^\circ\\ (1-r)(1-r')=b^2\end{array}\right\}\text{ and }\frac{1-r}{r}\Pi(-r)+\frac{1-r'}{r'}\Pi(-r')-\frac{c^2}{rr'}F=\int_0\frac{-dR}{1-rr'.R_2}$$

$$=\frac{-1}{2\sqrt{(rr')}}.\log.\frac{1+R\sqrt{(rr')}}{1-R\sqrt{(rr')}},$$

it being observed that Rrr' are all numerically less than 1.

If $r=c^2\sin^2\eta$, $r'=c^2\sin^2\eta^\circ$; and the logarithmic part is

$$\log\frac{1+c\sin\omega\sin\omega^\circ.\,c\sin\eta\sin\eta^\circ}{1-c\sin\omega\sin\omega^\circ.\,c\sin\eta\sin\eta^\circ}\text{ or }\log\frac{1+c_1\sin\omega_1\sin\eta_1}{1-c_1\sin\omega_1\sin\eta_1},$$

if we form c_1, ω_1, η_1 in Lagrange's scale from c, ω, η.

The same may take another form: viz. if $F\zeta=F\omega+F\eta$ and $F\varepsilon=F\omega-F\eta$, it becomes

$$\log\frac{\Delta\omega\Delta\eta+c^2\sin\omega\cos\omega\sin\eta\cos\eta}{\Delta\omega\Delta\eta-c^2\sin\omega\cos\omega\sin\eta\cos\eta}\text{ or }\log\frac{\Delta\varepsilon}{\Delta\zeta}.$$

Further, observe that

$$\sqrt{(rr')}\,\frac{1-r}{r}=\sqrt{T(-r)},\quad \sqrt{(rr')}\,\frac{1-r'}{r'}=\sqrt{T(-r')},$$

so that the general integral becomes

$$\sqrt{-T}\Pi(-r)+\sqrt{-T}\Pi(-r')=\frac{c^2}{\sqrt{(rr')}}F-\tfrac{1}{2}\log\frac{\Delta\varepsilon}{\Delta\zeta}\ldots(19).$$

In this, let $\omega=\frac{1}{2}\pi$, $\omega^\circ=0$, $R=0$, $\varepsilon=-\zeta$, therefore

$$\sqrt{-T}\Pi_c(-r)+\sqrt{-T}\Pi_c(-r')=\frac{c^2}{\sqrt{(rr')}}F_c\,\ldots\ldots..(19a).$$

Multiply the last by $\dfrac{F\omega}{F_c}$ and subtract the product from (19), therefore

$$\left.\begin{array}{l}\sqrt{-T}\mathrm{P}(-c^2\sin^2\eta)+\sqrt{-T}\mathrm{P}(-c^2\sin^2\eta^\circ)\\ =-\frac{1}{2}\log\dfrac{1+c_1\sin\omega_1\sin\eta_1}{1-c_1\sin\omega_1\sin\eta_1}=-\frac{1}{2}\log\dfrac{\Delta\varepsilon}{\Delta\zeta}\end{array}\right\}\ldots(20).$$

But the more important case is when Π is circular. Let

$$p=\cot^2\theta,\quad r=-1+b^2\sin^2\theta.$$

Observe that

$$\sqrt{(pr)}=\frac{p}{1+p}\sqrt{T(p)}=\frac{r}{1-r}\sqrt{T(-r)},$$

so that

$$\sqrt{T}.\Pi(p)-\sqrt{T}\Pi(-r)-\frac{c^2}{\sqrt{(pr)}}F=\tan^{-1}\{\sqrt{(pr)}\sin\omega\sin\omega^\circ\}\ldots(21).$$

We now see also that equation (6) admitted of being written

$$\sqrt{T}\{\Pi\omega+\Pi\omega^\circ-\Pi_c\}=\tan^{-1}\{\sqrt{(pr)}\sin\omega\sin\omega^\circ)\ldots(22),$$

where p is the common parameter: but the advantage of comparing the two last equations is best seen, when we have separated P out of Π. In (21), make $\omega=\frac{1}{2}\pi$, then

$$\sqrt{T}\Pi_c(p)-\sqrt{T}\Pi_c(-r)-\frac{c^2}{\sqrt{(pr)}}F_c=0.$$

Multiply by $\dfrac{F\omega}{F_c}$ and subtract from (21); then

$$\sqrt{T}\mathrm{P}(p)-\sqrt{T}\mathrm{P}(-r)=\tan^{-1}\{\sqrt{(pr)}\sin\omega\sin\omega^\circ\}\ldots(23).$$

Also subtract from (5) the equation

$$\sqrt{T}\left\{\frac{F\omega}{F_c}+\frac{F\eta}{F_c}-\frac{F\zeta}{F_c}\right\}\quad{}_c=0,$$

and it changes every Π in (5) into P. Observing, then, that $\mathrm{P}_o=0$, we have instead of (22) the simpler result

$$\sqrt{T}\{\mathrm{P}(p\omega)+\mathrm{P}(p\omega^\circ)\}=\tan^{-1}\{\sqrt{(pr)}\sin\omega\sin\omega^\circ\}\ldots(24).$$

Comparing then (23) and (24), we conclude that

$$\sqrt{T}.\mathrm{P}(-r,\omega)=-\sqrt{T}.\mathrm{P}(p,\omega^\circ)\ldots\ldots\ldots\ldots(25).$$

Hence the general enunciation: "In any circular $\sqrt{T}\mathrm{P}$ we may at pleasure change a parameter into its conjugate, provided that we change the amplitude also into the negative of its conjugate."

It is not difficult to verify (25) directly, by mere differentiation. Moreover, if we adapt the process to the case of a logarithmic Π, then, instead of (24), we get

$$\sqrt{-T}\{P(-r,\omega)+P(-r,\omega^\circ)\}=-\tfrac{1}{2}\log\frac{\Delta\varepsilon}{\Delta\zeta},$$

which, compared with (20), gives

$$\sqrt{-T}P(-c^2\sin^2\eta^\circ,\omega)=\sqrt{-T}P(-c^2\sin^2\eta,\omega^\circ)\ldots(25a).$$

Thus, "In a logarithmic $\sqrt{-T}P$ we may change the parameter into its conjugate, provided that we simultaneously change the amplitude into its conjugate. Or, "To get the integral conjugate to $\sqrt{-T}P(-c^2\sin^2\eta,\omega)$, we may at pleasure put η° for η, or ω° for ω."

Generally, even with a circular Π, it suffices to treat of *three* parameters, as p, its reciprocal q, and $-r$ the conjugate to p. But we may reckon *four* in the following method: first, $P(\cot^2\theta, c, \omega)$ its reciprocal $P(c^2\tan^2\theta, c, \omega)$; conjugate of the first, $-P(\cot^2\theta, c, \omega^\circ)$; conjugate of the second, $-P(c^2\tan^2\theta, c, \omega^\circ)$. And these, though in appearance four, are evidently in form only two. But for the present we shall continue to deal with three.

VII. Let us for conciseness write $\dfrac{F(c\omega)}{F_c}=\dfrac{x}{\frac{1}{2}\pi}$,

$$\Omega=\sqrt{TP}(pc\omega),\quad \overset{2}{\Omega}=\sqrt{TP}(qc\omega),\quad \overset{3}{\Omega}=\sqrt{TP}(-rc\omega),$$

where $p=\cot^2\theta$, $q=c^2\tan^2\theta=\cot^2\theta_\circ$, $r=\Delta^2(b\theta)$;

and there will be no danger of mistaking $\overset{2}{\Omega}$, $\overset{3}{\Omega}$ for *powers* of Ω, since no powers of Ω ever occur in any equation with which we deal.

Equations (18) and (23) now give the two results

$$\cot(x+\Omega+\overset{2}{\Omega})=\sin\theta\sin\theta_\circ\cdot\frac{b\cos\omega}{\cos\omega^\circ}\ \ldots\ldots(26),$$

$$\tan(\Omega-\overset{3}{\Omega})=\sin\omega\sin\omega^\circ\cdot\frac{c\cos\theta}{\cos\theta_\circ}\ \ldots\ldots\ldots(27);$$

which immediately suggest that if in (27) we commute $c\omega\theta$ into $b\theta\omega$, we make the members of the two equations identical.

Again, we may propose to ourselves to eliminate Ω between (26) and (27), or rather between (18) and (23), so as to get a relation between $\overset{2}{\Omega}$ and $\overset{3}{\Omega}$. This has been done by

Legendre, under a different notation; and the result is most unexpectedly simple.

For a moment, let

$$h^{-1} = \frac{\sin\theta\cos\theta}{\Delta(b\theta)}\cdot\frac{\Delta(c\omega)}{\tan\omega}, \qquad k = \frac{\sin\omega\cos\omega}{\Delta(c\omega)}\cdot\frac{\Delta(b\theta)}{\tan\theta};$$

or, if $D = \dfrac{\Delta(b\theta)\sin\omega}{\Delta(c\omega)\sin\theta}$, $h = \dfrac{D}{\cos\omega\cos\theta}$, $k = D\cos\omega\cos\theta$.

Also $\qquad x + \Omega + \overset{2}{\Omega} = \tan^{-1}h: \quad \Omega - \overset{3}{\Omega} = \tan^{-1}k;$

whence $\qquad x + \overset{2}{\Omega} + \overset{3}{\Omega} = \tan^{-1}h - \tan^{-1}k,$

or $\qquad \tan(x + \overset{2}{\Omega} + \overset{3}{\Omega}) = \dfrac{h-k}{1+hk}.$

Now $1 + hk = 1 + D^2 = \dfrac{\Delta^2(c\omega)\sin^2\theta + \Delta^2(b\theta)\sin^2\omega}{\Delta^2(c\omega)\sin^2\theta} = \dfrac{1-\cos^2\omega\cos^2\theta}{\Delta^2(c\omega)\sin^2\theta},$

and $\qquad h - k = h(1-\cos^2\omega\cos^2\theta);$

$$\therefore \frac{h-k}{1+hk} = h.\Delta^2(c\omega)\sin^2\theta = \frac{\Delta(b\theta)}{\cot\theta}\cdot\frac{\Delta(c\omega)}{\cot\omega} = \text{also } \frac{\sin\theta}{\sin\theta_\circ}\cdot\frac{\sin\omega}{\sin\omega^\circ}.$$

Thus we obtain between $\overset{2}{\Omega}$ and $\overset{3}{\Omega}$ the relation

$$\tan(x + \overset{2}{\Omega} + \overset{3}{\Omega}) = \frac{\sin\theta}{\sin\theta_\circ}\cdot\frac{\sin\omega}{\sin\omega^\circ} \quad\ldots\ldots\ldots (28).$$

The three equations (26), (27), (28) are a mere application of the *Reciprocal* and *Conjugates* of Legendre to the case of a circular $\sqrt{TP}$.

VIII. We now turn to the *Commutative* equations.

Making p to vary in Π, we get $\dfrac{d\Pi}{dp} = \displaystyle\int_0 \frac{-\sin^2\omega.dF}{(1+p\sin^2\omega)^2}$. By the general formula of reduction this receives the shape

$$\frac{d\Pi}{dp} = \alpha.\frac{\sin\omega\cos\omega\,\Delta\omega}{1+p\sin^2\omega} + \beta.F + \gamma.\int_0 \sin^2\omega\,dF + \delta.\Pi,$$

and it is found that $\alpha, \beta, \gamma, \delta$ have a common denominator, which is none other than $T(p)$. Calling the numerators $\alpha', \beta', \gamma', \delta'$, we find

$$\alpha' = \tfrac{1}{2}, \quad \beta' = -\tfrac{1}{2}.\frac{c^2}{p^2}, \quad \gamma' = -\tfrac{1}{2}.\frac{c^2}{p}, \quad \delta' = -\tfrac{1}{2}.\left(1 - \frac{c^2}{p^2}\right),$$

which farther yield

$$\delta' = -\tfrac{1}{2}\cdot\frac{dT}{dp},\quad \beta' = \tfrac{1}{2}\cdot\left(\frac{dT}{dp} - 1\right).$$

Observing then that $c^2 \int_0 \sin^2\omega . dF = F - E$, we get

$$T.\frac{d\Pi}{dp} = \tfrac{1}{2}.\frac{\sin\omega\cos\omega\Delta\omega}{1 + p\sin^2\omega} + \tfrac{1}{2}.\left(\frac{dT}{dp} - 1\right)F - \frac{1}{2p}(F - E) - \tfrac{1}{2}.\frac{dT}{dp}.\Pi.$$

Bringing all the T's and Π's to the left, we divide either by $\sqrt{T}$ or by $-\sqrt{-T}$ to make the left-hand an exact integral, and then integrate nearly as Legendre.

It gives

$$\left.\begin{aligned} &\sqrt{T}(\Pi - F) \\ &= \tfrac{1}{2}\sin\omega\cos\omega\Delta\omega.\int\frac{\sqrt{T^{-1}}dp}{1+p\sin^2\omega} - \tfrac{1}{2}F\int\left(1+\frac{1}{p}\right)\frac{dp}{\sqrt{T}} + \tfrac{1}{2}E\int\frac{dp}{p\sqrt{T}} \\ &\sqrt{-T}(\Pi - F) \\ &= -\tfrac{1}{2}\sin\omega\cos\omega\Delta\omega.\int\frac{\sqrt{-T^{-1}}dp}{1+p\sin^2\omega} + \tfrac{1}{2}F\int\left(1+\frac{1}{p}\right)\frac{dp}{\sqrt{-T}} - \tfrac{1}{2}E\int\frac{dp}{p\sqrt{-T}} \end{aligned}\right\}..(29),$$

in which the integrals may begin from $p = 0$, since this supposition makes the left-hand member vanish, and also involves no infinite quantities, such as would appear in Legendre's equation.

In the last let $\omega = \frac{1}{2}\pi$;

$$\therefore\ \sqrt{-T}(\Pi_c.p - F_c) = \tfrac{1}{2}F_c.\int_0\left(1 + \frac{1}{p}\right)\frac{dp}{\sqrt{-T}} - \tfrac{1}{2}E_c\int_0\frac{dp}{p\sqrt{-T}}\ldots(29a).$$

Now, if $\quad p = -c^2\sin^2\eta,\quad \sqrt{-T} = \cot\eta\Delta(c\eta),$

whence $\frac{1}{2}.\left(1 + \frac{1}{p}\right)\frac{dp}{\sqrt{-T}} = \Delta(c\eta)\,d\eta,\quad \frac{1}{2}.\frac{dp}{p\sqrt{-T}} = \frac{d\eta}{\Delta(c\eta)},$

$$\text{or}\quad \left.\begin{aligned}\sqrt{-T}\{\Pi_c(-c^2\sin^2\eta) - F_c\} &= F_cE(c\eta) - E_cF(c\eta) \\ &= F_cG(c\eta)\end{aligned}\right\}\ldots(30).$$

We may now apply equation (13) to save the trouble of a second integration for the case of the circular Π. It is only requisite to put $\sin\eta = \sqrt{-1}\tan\theta$, and there follows (multiplying or dividing by $\sqrt{-1}$),

$$\sqrt{T}\{F_c - \Pi_c(c^2\tan^2\theta)\} = -F_c\{H(b\theta) - \Delta(b\theta)\tan\theta\}\ldots(31).$$

There is ambiguity as to the sign of $\sqrt{T}$, which is changed according as we *multiply* or *divide* by $\sqrt{-1}$. But we determine the sign by observing that when p is positive, $F > \Pi$; also when θ approaches $\frac{1}{2}\pi$, evidently $\Delta(b\theta)\tan\theta > H(b\theta)$, since $F_cH_b = \frac{1}{2}\pi$.

We may deduce $\Pi_c(-1+b^2\sin^2\theta)$ by combining the last with (21) and with (17*a*).

From 21,

$$\sqrt{T}\Pi_c(p)-\sqrt{T}\Pi_c(-r)=\frac{c^2}{\sqrt{(pr)}}F_c.$$

Also $\quad \sqrt{T}\{\Pi_c p+\Pi_c q-F_c\}=\frac{1}{2}\pi$ from (17*a*);

therefore $\sqrt{T}\{\Pi_c(q)-F_c\}+\sqrt{T}\Pi_c(-r)=\frac{1}{2}\pi-\frac{c^2}{\sqrt{(pr)}}F_c,$

or $\sqrt{T}\{\Pi_c(-r)-F_c\}-\sqrt{T}\{F_c-\Pi_c(q)\}=\frac{1}{2}\pi-\left\{\sqrt{T(-r)}+\frac{c^2}{\sqrt{(pr)}}\right\}F_c$

$$=\tfrac{1}{2}\pi-\Delta(b\theta)\tan\theta.F_c,\quad \text{if } p=\cot\theta.$$

Add the last to (31), and it gives

$$\sqrt{T}\{\Pi_c(-\Delta^2.b,\theta)-F_c\}=\tfrac{1}{2}\pi-F_c.H(b\theta)\dots(32).$$

This completes the equation needed for the integral Π_c, which is entirely reduced to F and E.

We vary the form only, by adding (17*a*) to (31);

therefore $\quad \sqrt{T}\Pi_c(p)=\frac{1}{2}\pi-F_c\{H(b\theta)-\Delta(b\theta)\tan\theta\}$;

subtract it from $\sqrt{T}.F_c=\dfrac{\Delta(b\theta)}{\sin\theta\cos\theta}.F_c$;

then $\sqrt{T}\{F_c-\Pi_c(\cot^2\theta)\}=F_c\{H(b\theta)+\Delta(b\theta)\cot\theta\}-\frac{1}{2}\pi\dots(33).$

We may also develop the right-hand member of equation (30) by Lagrange's scale; then

$$\sqrt{-T}.\{\Pi_c(-c^2\sin^2\eta)-C\}=C_1c_1\sin\eta_1+C_2c_2\sin\eta_2+C_3c_3\sin\eta_3+\&c\dots;$$

and if we then assume $-c_n{}^2\sin^2\eta_n=p_n$,

$$\left.\begin{aligned}\sqrt{-T}.\{\Pi_c-C\}&=C_1\sqrt{-p_1}+C_2\sqrt{-p_2}+C_3\sqrt{-p_3}+\&c\dots\\ \therefore\ \sqrt{T}.\{C-\Pi_c\}&=C_1\sqrt{p_1}+C_2\sqrt{p_2}+C_3\sqrt{p_3}+\&c\dots\end{aligned}\right\}\dots(30a),$$

if we suppose p positive in the last. But to this subject we shall recur.

Π_c being now fully known, it remains to investigate P only, and Π will be known.

We must return to equation (29). If we multiply (29*a*) by $\dfrac{F(c\omega)}{F_c}$ and combine it with (29), we get

$$\sqrt{-T}P=-\tfrac{1}{2}\sin\omega\cos\omega\Delta(c\omega)\int_0\frac{\sqrt{-T^{-1}}dp}{1+p\sin^2\omega}-G(c\omega)\int_0\frac{dp}{2p\sqrt{-T}}.$$

We have already found the last integral = $F(c\eta)$, when $p = -c^2 \sin^2\eta$. Also

$$\frac{\sqrt{-T^{-1}}.dp}{1+p\sin^2\omega} = \frac{2c^2 \sin\eta \cos\eta\, d\eta}{(1-c^2\sin^2\omega\sin^2\eta)\cot\eta\,\Delta(c\eta)} = \frac{-2c^2.\sin^2\eta.dF(c\eta)}{1-c^2\sin^2\omega\sin^2\eta}$$

$$= \frac{2}{\sin^2\omega}\left\{1 - \frac{1}{1-c^2\sin^2\omega\sin^2\eta}\right\} dF(c\eta);$$

therefore $\sqrt{-T}P(-c^2\sin^2\eta, c, \omega)$

$$= \frac{\Delta(c\omega)}{\tan\omega}\{\Pi(-c^2\sin^2\omega, c, \eta) - F(c\eta)\} - G(c\omega).F(c\eta),$$

where we may write simply $\sqrt{-T}$ for the multiplier of the quantity in brackets.

Again, in (30) change η into ω, and multiply by $F(c\eta) \div F_c$;

therefore $0 = \sqrt{-T}.\{\Pi_c(-c^2\sin^2\omega) - F_c\}\dfrac{F(c\eta)}{F_c} - G(c\omega)F(c\eta)$:

subtract this from the preceding; then

$$\sqrt{-T}P(-c^2\sin^2\eta, c, \omega) = \sqrt{-T}P(-c^2\sin^2\omega, c, \eta)\ldots(34);$$

this is the commutative equation for a logarithmic P.

For a circular P the result is not quite so simple. We have first

$$\sqrt{T}P = \sin\omega\cos\omega\,\Delta(c\omega)\int_0 \frac{\frac{1}{2}\sqrt{T^{-1}}.dp}{1+p\sin^2\omega} + G(c\omega)\int_0 \frac{dp}{2p\sqrt{T}}.$$

Let $$p = \cot^2\theta;$$

$$\therefore \frac{dp}{2p} = \frac{-d\theta}{\sin\theta\cos\theta},\quad \sqrt{T} = \frac{\Delta(b\theta)}{\sin\theta\cos\theta},\quad \frac{dp}{2p\sqrt{T}} = \frac{-d\theta}{\Delta(b\theta)} = -dF(b\theta).$$

Also

$$\frac{1}{2\sqrt{T}}.\frac{dp}{1+p\sin^2\omega} = \frac{-\cot^2\theta\, d\theta}{\Delta(b\theta)(1+\sin^2\omega\cot^2\theta)} = \frac{-\cos^2\theta.dF(b\theta)}{\sin^2\theta+\sin^2\omega.\cos^2\theta}.$$

The denominator $= \sin^2\omega + \cos^2\omega\sin^2\theta = \sin^2\omega(1+\cot^2\omega\sin^2\theta)$;

$$\therefore \int\frac{1}{2\sqrt{T}}.\frac{dp}{1+p\sin^2\omega} = \frac{F(b\theta)}{\cos^2\omega} - \frac{\Pi(\cot^2\omega, b, \theta)}{\sin^2\omega\cos^2\omega} + \text{const.}$$

Instead of correcting by making $\theta = \frac{1}{2}\pi$, $p = 0$, we may make $p = \infty$, $\theta = 0$; in which case we know that $\sqrt{T}\Pi(p) = \frac{1}{2}\pi$ for all values of ω, and consequently

$$\sqrt{T}P(p, c, \omega) = \tfrac{1}{2}\pi\left\{1 - \frac{F(c\omega)}{F_c}\right\} = \tfrac{1}{2}\pi - x;$$

$$\therefore \sqrt{T}P(\cot^2\theta, c, \omega) = \sin\omega\cos\omega\,\Delta(c\omega)\left\{\frac{F(b\theta)}{\cos^2\omega} - \frac{\Pi(\cot^2\omega, b, \theta)}{\sin^2\omega\cos^2\omega}\right\}$$
$$- G(c\omega)F(b\theta) + (\tfrac{1}{2}\pi - x).$$

Now put $\theta = \frac{1}{2}\pi$, and the left-hand member vanishes;

or $0 = \sin\omega\cos\omega\Delta(c\omega)\left\{\frac{F_b}{\cos^2\omega} - \frac{\Pi_b(\cot^2\omega)}{\sin^2\omega\cos^2\omega}\right\} - G(c\omega)F_b + (\frac{1}{2}\pi - x)$.

Multiply the last by $\frac{F(b\theta)}{F_b} = \frac{t}{\frac{1}{2}\pi}$, and subtract from the penultimate; observing that $\frac{\Delta(c\omega)}{\sin\omega\cos\omega} = \sqrt{T}(\cot^2\omega)$; hence

$$\sqrt{T}P(\cot^2\theta, c, \omega) + \sqrt{T}P(\cot^2\omega, b, \theta) = (\tfrac{1}{2}\pi - x)\left(1 - \frac{t}{\frac{1}{2}\pi}\right) \ldots (35),$$

in which the right-hand member could not have been conjectured from the analogy of (34). It has risen from the peculiarity that $\sqrt{T}\Pi(p) = \frac{1}{2}\pi$ when $p = \infty$.

IX. We may now revert to the notation of (26), (27), (28), by writing Θ for that which Ω becomes when $\theta c\omega$ are changed to $\omega b\theta$. Then, in place of (35), we get

$$\tfrac{1}{2}\pi\left\{\Omega + \Theta\right\} = (\tfrac{1}{2}\pi - x)(\tfrac{1}{2}\pi - t) \ldots\ldots\ldots (36).$$

Between the quantities $\Omega, \overset{2}{\Omega}, \overset{3}{\Omega}, \Theta, \overset{2}{\Theta}, \overset{3}{\Theta}$ we have obtained three independent equations (26), (27), (36). If we commute θ, c, ω with ω, b, θ, (36) is not changed, but (26) and (27) yield two new equations, so that we have, in all, *five* equations connecting the *six* quantities in pairs. We have, therefore, exhausted all the relations between them; and by mere elimination we can obtain the equation between any pair of them at pleasure.

In a single view the equations are as follows:

$$\left.\begin{aligned} &\cot(x + \Omega + \overset{2}{\Omega}) = \tan(\Theta - \overset{3}{\Theta}) \\ &= \tan\left\{\frac{t(\frac{1}{2}\pi - x)}{\frac{1}{2}\pi} + \Theta - \overset{2}{\Omega}\right\} = \tan\left\{\frac{(\frac{1}{2}\pi - x)(\frac{1}{2}\pi - t)}{\frac{1}{2}\pi} - \Omega - \overset{3}{\Theta}\right\} \\ &= \sin\theta\sin\theta_\circ \cdot \frac{b\cos\omega}{\cos\omega^\circ} \end{aligned}\right\} \ldots (37),$$

in which we may exchange $\theta, c, \omega, \Omega, x$ with $\omega, b, \theta, \Theta, t$.

$$\left.\begin{aligned} &\text{Again,}\ \tan(x + \overset{2}{\Omega} + \overset{3}{\Omega}) = \tan(t + \overset{2}{\Theta} + \overset{3}{\Theta}) \\ &= \tan\left\{\frac{xt}{\frac{1}{2}\pi} + \overset{2}{\Omega} + \overset{3}{\Theta}\right\} = \cot\left\{\frac{(\frac{1}{2}\pi - x)(\frac{1}{2}\pi - t)}{\frac{1}{2}\pi} - \overset{3}{\Omega} - \overset{3}{\Theta}\right\} \\ &= \frac{\sin\theta}{\sin\theta_\circ} \cdot \frac{\sin\omega}{\sin\omega^\circ} \end{aligned}\right\} \ldots (38).$$

Lastly,

$$\left.\begin{aligned} \tfrac{1}{2}\pi(\Omega + \Theta) &= (\tfrac{1}{2}\pi - x)(\tfrac{1}{2}\pi - t) \\ \tfrac{1}{2}\pi(\overset{2}{\Theta} - \overset{3}{\Omega}) &= x(\tfrac{1}{2}\pi - t) \\ \tfrac{1}{2}\pi(\overset{2}{\Omega} - \overset{3}{\Theta}) &= t(\tfrac{1}{2}\pi - x) \end{aligned}\right\} \text{.........(39).}$$

In (37) there are, with the commutations, 8 equations; in (38) there are 4; so that we have here 15 relations of pairs of the six integrals. Also $15 = \dfrac{5.6}{2}$; so that all the possible pairs are here exhibited.

Thus by a single transformation we can pass from one circular $\sqrt{TP}$ to another, so as to secure at once that p shall be less than q, (and therefore, p less than c), and that the modulus shall be less than its complement. We may also, if we wish it, secure a positive parameter.

X. Now rises the question, of approximating to $\sqrt{TP}$, when we have so transformed it as to obtain a favourable case. We proceed by Lagrange's scale, supposing $c_1\omega_1$ to be deduced as usual from $c\omega$. Let p also be given, but $p_1 r$ be disposable constants. Then if Π_1 stand for $\Pi(p_1 c_1 \omega_1)$, and we make the usual substitutions $F(c_1\omega_1) = (1+b)\,F(c\omega)$, $\sin\omega_1 = (1+b)\dfrac{\sin\omega\cos\omega}{\Delta(c\omega)}$, we get

$$\Pi_1 = \int_0 \frac{dF(c_1\omega_1)}{1 + p_1\sin^2\omega_1} = (1+b)\int_0 \frac{\Delta^2(c\omega)\,dF(c\omega)}{\Delta^2(c\omega) + p_1(1+b)^2\sin^2\omega\cos^2\omega}.$$

We determine the relations between p_1, p, and r, if we assume the denominator to be $= (1 + p\sin^2\omega)(1 - r\sin^2\omega)$; which gives $pr = p_1(1+b)^2$, and making $\omega = \frac{1}{2}\pi$, $(1+p)(1-r) = b^2$; so that $-r$ is none other than the parameter conjugate to p.

Also, $$p_1 = \frac{pr}{(1+b)^2}, \quad q_1 = \frac{(1-b)^2}{pr}.$$

Hence $$T_1 = (1+p_1)(1+q_1) = \frac{(\overline{1+b}|^2 + pr)(\overline{1-b}|^2 + pr)}{(1+b)^2.pr},$$

of which the numerator

$$= c^4 + 2(1+b^2)pr + p^2r^2 = (pr - c^2)^2 + 4pr = (p-r)^2 + 4pr = (p+r)^2;$$

therefore $$\sqrt{\frac{T_1}{pr}} = \frac{p+r}{(1+b)\,pr}.$$

Thus T_1 is positive, when pr is positive, or $T_1\ T$ are both circular or both logarithmic.

Farther, if we assume

$$\frac{1-c^2v}{(1+pv)(1-rv)}=\frac{M}{1+pv}+\frac{N}{1-rv},$$

which yields

$$\Pi_1=(1+b)\{M\Pi(p)+N\Pi(-r)\},$$

we get $M+N=1$, $Mr-Np=c^2$; whence

$$M=\frac{p+c^2}{p+r}=\frac{1+p}{p}\cdot\left(\frac{pr}{p+r}\right), \text{ and } N=\frac{r-c^2}{p+r}=\frac{1-r}{r}\cdot\left(\frac{pr}{p+r}\right),$$

whence $\Pi_1=(1+b)\cdot\frac{pr}{p+r}\left\{\frac{1+p}{p}\Pi(p)+\frac{1-r}{r}\Pi(-r)\right\}$.

Multiply by $\sqrt{T_1}=\frac{p+r}{(1+b)\sqrt{(pr)}}$;

therefore $\sqrt{T_1}\,\Pi_1=\sqrt{T}\,\Pi(p)+\sqrt{T}\,\Pi(-r)$.........(40).

Make $\omega=\frac{1}{2}\pi$, $\omega_1=\pi$; $\therefore 2\sqrt{T_1}\Pi_{c1}=\sqrt{T}\Pi_c(p)+\sqrt{T}\Pi_c(-r)$.

Multiply this by $\frac{1}{2}\cdot\frac{F_1}{C_1}=\frac{F}{C}$, and subtract from (40),

then $\sqrt{\pm\, T_1}\mathrm{P}_1=\sqrt{\pm\, T}\mathrm{P}(p)+\sqrt{\pm\, T}\mathrm{P}(-r)$......(40$a$),

which for the circular integral is

$$\Omega_1=\Omega+\overset{3}{\Omega}\text{ .... (41).}$$

But from (27), observing that

$$\sin\omega\,\sin\omega^\circ.\frac{c\cos\theta}{\cos\theta_\circ}=\frac{\sin\omega_1}{1+b}.\sqrt{(pr)}=\sqrt{p_1}\,\sin\omega_1,$$

we have $\Omega-\overset{3}{\Omega}=\tan^{-1}(\sqrt{p_1}\,\sin\omega_1)$.

Add this to (41), and you get

$$\Omega=\tfrac{1}{2}\Omega_1+\tfrac{1}{2}\tan^{-1}(\sqrt{p_1}\,\sin\omega_1)\text{............(42):}$$

but in the process there has been nothing up to (40a) to limit p to be circular. When otherwise, it is very obvious how to change $\tan^{-1}$ into $\sqrt{-1}.\log$ in the equation (42) so as to adapt it as in equation (20).

XI. Legendre has investigated the relations of p_1 and p; but we need to add many reflections.

When $p=-c^2\sin^2\eta$, $r=c^2\sin^2\eta^\circ$,

$$\sqrt{(-pr)}=c^2\sin\eta\,\sin\eta^\circ=\frac{c^2}{1+b}\sin\eta_1;$$

therefore $\sqrt{-p_1} = \dfrac{c^2}{(1+b)^2} \sin\eta_1 = c_1 \sin\eta_1$.

Consequently the series p, p_1, p_2...answers the conditions supposed in equation (30a), and those developments apply to our newly-derived parameters.

When we assume $\sin\eta = \sqrt{-1}\tan\theta$, or $p = c^2\tan^2\theta$, the general equation of Lagrange's scale, viz. $\tan\eta_1 = \dfrac{(1+b)\tan\eta}{1-b\tan^2\eta}$, changes into the scale of Gauss, $\sin\theta_1 = \dfrac{(1+b)\sin\theta}{1+b\sin^2\theta}$, which is that followed by $\theta, \theta_1, \theta_2, \theta_3$... Consequently, when $\theta = \frac{1}{2}\pi$, every other θ in the scale also $= \frac{1}{2}\pi$. Also θ, θ_1, θ_2, θ_3... tend to $\frac{1}{2}\pi$ as their limit, since they give

$$\frac{F(b\theta)}{B} = \frac{F(b_1\theta_1)}{B_1} = \frac{F(b_2\theta_2)}{B_2};$$

and since $B_n = \infty$ when $n = \infty$, and $b_n = 1$, therefore

$$F(1\theta_n) = \infty, \text{ or } \theta_n = \tfrac{1}{2}\pi.$$

If we form θ, θ', θ''... in the opposite direction, then, since

$$\frac{t}{\frac{1}{2}\pi} = \frac{F(b\theta)}{B} = \frac{F(b'\theta')}{B'} = \frac{F(b''\theta'')}{B''} = \&\text{c}...$$

and $B^{(n)}$ has $\frac{1}{2}\pi$ for limit, and $b^{(n)}$ is evanescent, therefore $\theta^{(n)} = t$, when $n = \infty$; or θ, θ', θ'', θ'''... tend to t as their limit.

It is farther important to observe, that the change of $b\theta$ to $b_n\theta_n$ or to $b^{(n)}\theta^{(n)}$ leaves t wholly unchanged; inasmuch as

$$\frac{t}{\frac{1}{2}\pi} = \frac{F(b\theta)}{B} = \frac{F(b_n\theta_n)}{B_n} = \frac{F(b^{(n)}\theta^{(n)})}{B^{(n)}}.$$

On the other hand, it is familiar, as a result of the equation in Lagrange's scale,

$$\frac{x}{\frac{1}{2}\pi} = \frac{F(c\omega)}{C} = \tfrac{1}{2}\cdot\frac{F(c_1\omega_1)}{C_1} = \tfrac{1}{4}\cdot\frac{F(c_2\omega_2)}{C_2} = \&\text{c}...,$$

that to change $c\omega$ into $c_1\omega_1$ changes x into $2x$, &c.

The equation $\sin\theta_1 = \dfrac{(1+b)\sin\theta}{1+b\sin^2\theta}$, which gives $p_1 = c_1^{\,2}\tan^2\theta_1$, admits of other forms; especially

$$\cos\theta_1 = \frac{\cos\theta.\Delta(b\theta)}{1+b\sin^2\theta}, \quad \Delta(b_1\theta_1) = \frac{1-b\sin^2\theta}{1+b\sin^2\theta};$$

which last gives inversely

$$b \sin^2\theta = \frac{1 - \Delta(b_1\theta_1)}{1 + \Delta(b_1\theta_1)}.$$

Legendre calculates $p, p_1, p_2 \ldots$ by auxiliary arcs $\lambda, \lambda_1, \lambda_2, \lambda_3 \ldots$ Let $\cos\lambda = \Delta(b\theta)$, $\cos\lambda_1 = \Delta(b_1\theta_1)$, &c.; therefore $\sin\lambda = b\sin\theta$. Consequently $\frac{\sin^2\lambda}{b} = \frac{1-\cos\lambda_1}{1+\cos\lambda_1}$, or $\sin\lambda = \sqrt{b}\,.\tan\frac{1}{2}\lambda_1$; which is a general relation for $\lambda, \lambda_1, \lambda_2, \lambda_3 \ldots$

The arcs $\lambda, \lambda_1, \lambda_2 \ldots$ are thus suggested by the parameter $\cot^2\theta$, but they equally apply to logarithmic parameters. In fact, if we assume

$$\sin\lambda = \frac{b}{\Delta(c\eta)}, \quad \sin\lambda_1 = \frac{b_1}{\Delta(c_1\eta_1)},$$

and observe that $\Delta^2(c\eta) = b^2 + c^2\cos^2\eta$, we get

$$\cos\lambda = \frac{c\cos\eta}{\Delta(c\eta)}, \quad \cos\lambda_1 = \frac{c_1\cos\eta_1}{\Delta(c_1\eta_1)}.$$

Now, by the known relations in Lagrange's scale,

$$\cos\eta_1 = \frac{1-(1+b)\sin^2\eta}{\Delta(c\eta)}, \quad \text{and } \Delta(c_1\eta_1) = \frac{1-(1-b)\sin^2\eta}{\Delta(c\eta)},$$

and

$$c_1 = \frac{1-b}{1+b};$$

whence $$\cos\lambda_1 = \frac{1-b}{1+b}\cdot\frac{1-(1+b)\sin^2\eta}{1-(1-b)\sin^2\eta} = \frac{1-b-c^2\sin^2\eta}{1+b-c^2\sin^2\eta},$$

and $$\frac{1-\cos\lambda_1}{1+\cos\lambda_1} = \frac{b}{1-c^2\sin^2\eta}, \quad \text{or } \tan\tfrac{1}{2}\lambda_1 = \frac{\sin\lambda}{\sqrt{b}},$$

as before. Nevertheless, if λ has the same value in both cases, the relation of θ to η is no longer $\sin^2\eta = -\tan^2\theta$, but is $c^2\sin^2\eta = -\cot^2\theta$.

It thus appears that the series $p_1, p_2, p_3 \ldots$ which are derived from p by the law $p_1 = \frac{pr}{(1+b)^2}$, are calculable for both kinds of integral by the auxiliaries $p = \cot^2\theta = -c^2\sin^2\eta$,

$$\tan\tfrac{1}{2}\lambda_1 = \frac{\sqrt{b}}{\sqrt{(1+p)}}, \quad \tan\tfrac{1}{2}\lambda_2 = \frac{\sin\lambda_1}{b_1}, \quad \tan\tfrac{1}{2}\lambda_3 = \frac{\sin\lambda_2}{b_2}, \text{ \&c.} \ldots$$

provided only that $1 + p$ be positive.

Put for a moment $\cos\lambda_r = x$, $\cos\lambda_{r+1} = y$; $\therefore \frac{1-y}{1+y} = \frac{1-x^2}{b_r}$; and since $b_1, b_2, b_3 \ldots$ rapidly tend to 1, the last equation tends

to give $y = \frac{x^2}{2-x^2}$, or scarcely more than $y = \frac{1}{2}x^2$. Thus $\cos\lambda_1$, $\cos\lambda_2$, $\cos\lambda_3$... soon tend rapidly to zero.

These arcs enable us to embrace in one expression the two series (30*a*).

Let N for a moment stand for the complete integral $\int_0^{\frac{1}{2}\pi} \frac{(1+p)\sin^2\omega}{1+p\sin^2\omega} . dF(c\omega)$, so that $\frac{pN}{1+p} = F_c - \Pi_c$; and when $p = -c^2\sin^2\eta$, $\sqrt{-T} = \cot\eta\,\Delta(c\eta)$, and

$$\sqrt{-T}\{\Pi_c - F_c\} = \frac{-p}{1+p}.\cot\eta\,\Delta(c\eta).N = \frac{c^2\sin\eta\cos\eta}{\Delta(c\eta)}.N = (1-b)\sin\eta_1.N.$$

Consequently, from equation (30) we deduce two forms,

$$(1-b)\sin\eta_1.N = CG(c\eta); \quad\text{and}\quad \frac{c_1^2\sin\eta_1\cos\eta_1}{\Delta(c_1\eta_1)}.N_1 = C_1G(c_1\eta_1):$$

but $CG(c\eta) - C_1G(c_1\eta_1) = C_1c_1\sin\eta_1$. Combine these, and observe that $(1-b)\,C = 2C_1c_1$; and you get

$$\frac{N}{C} - \tfrac{1}{2}\cos\lambda_1.\frac{N_1}{C_1} = \tfrac{1}{2}\ldots\ldots\ldots\ldots(43),$$

which now applies alike to both sorts of integrals, and easily gives (when $1+p$ is positive)

$$N = \int_0^{\frac{1}{2}\pi}\frac{(1+p)\sin^2\omega}{1+p\sin^2\omega}\,dF$$

$$= C\{\tfrac{1}{2} + \tfrac{1}{4}\cos\lambda_1 + \tfrac{1}{8}\cos\lambda_1\cos\lambda_2 + \tfrac{1}{16}\cos\lambda_1\cos\lambda_2\cos\lambda_3 + \&c\ldots\}\ldots(43a),$$

which is a series of Legendre's slightly simplified.

XII. Write ψ instead of θ_0, so that $\psi, \psi_1, \psi_2, \psi_3$... may be (relatively to c, c_1, c_2, c_3...) the lower conjugates to $\theta, \theta_1, \theta_2, \theta_3$... Then if $p_nq_n = c_n^2$, we have simultaneously $p_n = c_n^2\tan^2\theta_n = \cot^2\psi_n$ and $q_n = c_n^2\tan^2\psi_n = \cot^2\theta_n$. Now, *I say*, $\psi, \psi_1, \psi_2, \psi_3$... are related to one another by the scale of Gauss, exactly as are $\theta, \theta_1, \theta_2, \theta_3$...

For we have, by definition,

$$F(b\theta) + F(b\psi) = B \quad\text{and}\quad F(b_1\theta_1) + F(b_1\psi_1) = B_1:$$

also, by the property of $\theta\theta_1$, we have

$$\frac{F(b\theta)}{B} = \frac{F(b_1\theta_1)}{B_1};\ \text{that is}\ \left\{1 - \frac{F(b\psi)}{B}\right\} = \left\{1 - \frac{F(b_1\psi_1)}{B_1}\right\},$$

or $\frac{F(b\psi)}{B} = \frac{F(b_1\psi_1)}{B_1}$:

which proves $\psi\psi_1$ to be related by the scale of Gauss.

Consequently, whether we assume $p = c^2 \tan^2\theta$, $p_1 = c_1^2 \tan^2\theta_1$, &c....; or, on the other hand, assume $p = \cot^2\theta$, $p_1 = \cot^2\theta_1$, &c.... the series $\theta, \theta_1, \theta_2 ...$ which determine $p, p_1, p_2 ...$ are deduced by the very same law. It amounts to the same thing to remark, that we have

$$p_1 = \frac{p^2}{(1+b)^2} \cdot \frac{1+q}{1+p}; \quad q_1 = \frac{q^2}{(1+b)^2} \cdot \frac{1+p}{1+q};$$

so that q_1 is formed from q, by the same law as p_1 from p.

Thus, "When two original parameters are reciprocals, in reference to the original modulus, so are every derived pair, in reference to the new modulus."

It also follows, that the more rapid the convergence of $p, p_1, p_2, p_3, p_4 ...$, the slower is the convergence of $q, q_1, q_2, q_3 ...$ if indeed they converge at all. This makes it important whether p or q be selected to approximate from. Legendre's equation is

$$p_1 = \frac{p}{1+p} \cdot \frac{p+c^2}{(1+b)^2};$$

whence, if $p = ec$, and $p_1 = e_1 c_1$, there follows

$$\frac{e_1}{e} = \frac{e+c}{1+ec}, \text{ or } 1 \pm \frac{e_1}{e} = \frac{(1 \pm c)(1 \pm e)}{1+ec}.$$

If then $p^2 < c^2$, $e^2 < 1$, and we deduce that $\left(1 \pm \frac{e_1}{e}\right)$ is positive, or e_1 is numerically less than e. Hence if $p^2 < c^2$, $p, p_1, p_2, p_3 ...$ decrease more rapidly than $c, c_1, c_2, c_3 ...$

The relation of p_1 to p may also be written

$$\sqrt{p_1} = \frac{p}{1+p} \cdot \frac{\sqrt{T}}{1+b};$$

but our original equation, $\sqrt{p_1} = \dfrac{\sqrt{(pr)}}{1+b}$, with equal clearness shews that p_1 is positive whenever p is circular. Nor only so. It also denotes that if the original integral be $\Pi(-r, c, \omega)$, the value of p_1 is not altered; for to change p to $-r$, does but change r to $-p$, and leaves (pr) unchanged. Nevertheless, a consideration of the process which elicited equation (42), shews that the conjugates $p, -r$ in the first step downwards generate $+\sqrt{p_1}$ and $-\sqrt{p_1}$ differing in sign: but in the second step they coincide, and both produce the same $\sqrt{p_2}$.

When the problem is reversed, and we desire from p_1 to determine p, it is evident that there are two roots, p and $-r$,

positive and *negative*, when Π is circular. But we cannot carry the series backward from $-r$, without falling on imaginary parameters. In fact, as p_1 is positive, if it proceed from a real p and r, so $-r$, being negative, cannot proceed from a real p' and r'. It may here deserve remark, that we thus learn of *certain* imaginary parameters, whose integral can be reduced to *one* real Π.

XIII. It remains to develop the actual series.

Let $\Psi = \tan^{-1}(p \sin \omega)$, so that

$$\Omega = \tfrac{1}{2}\Omega_1 + \tfrac{1}{2}\Psi_1.$$

When p is evanescent, $\Pi_c = F_c$, and P becomes identical with $\Pi - F$. We have seen that $\sqrt{T}(\Pi - F)$ vanishes with p; so therefore does $\sqrt{T}\text{P}$. If then $p, p_1, p_2, p_3 \ldots$ decrease beyond all limit, so do $\Omega, \Omega_1, \Omega_2, \Omega_3 \ldots$; and much more does $2^{-n}\Omega_n$ vanish when $n = \infty$. Now, by repetition of the formula,

$$\Omega - 2^{-n}\Omega_n = \tfrac{1}{2}\Psi_1 + \tfrac{1}{4}\Psi_2 + \tfrac{1}{8}\Psi_3 + \ldots + 2^{-n}\Psi_n;$$

whence $$\Omega = \tfrac{1}{2}\Psi_1 + \tfrac{1}{4}\Psi_2 + \tfrac{1}{8}\Psi_3 + \text{\&c. \&c.}\ldots,$$

which is the final development by *descending* moduli.

In the other notation we have

$$\left.\begin{aligned}\sqrt{T}\text{P}(\ p, c, \omega) &= \tfrac{1}{2}\tan^{-1}(\sqrt{p_1}\sin\omega_1) + \tfrac{1}{4}\tan^{-1}(\sqrt{p_2}\sin\omega_2) \\ &\quad + \tfrac{1}{8}\tan^{-1}(\sqrt{p_3}\sin\omega_3) + \text{\&c.} \\ \sqrt{T}\text{P}(-r, c, \omega) &= -\tfrac{1}{2}\tan^{-1}(\sqrt{p_1}\sin\omega_1) + \tfrac{1}{4}\tan^{-1}(\sqrt{p_2}\sin\omega_2) \\ &\quad + \tfrac{1}{8}\tan^{-1}.\ \text{\&c.}\ldots\end{aligned}\right\}\ldots(44).$$

Taking the difference of these,

$$\Omega - \overset{3}{\Omega} = \tan^{-1}(\sqrt{p_1}\sin\omega_1);$$

which agrees with (27).

The worst convergence is when $p = c$: and since we may select of c and b the smaller, by means of the commutative equation, the most unfavourable case which needs to be encountered, is, when $p = c = b$. Even then, the series (44) is computable with very moderate trouble. I conclude therefore that it is really sufficient for practical purposes.

Nevertheless, when c is near to 1, we may seek for a development by means of *ascending* moduli. We shall select $\overset{2}{\Theta} = \sqrt{T}\text{P}(b^2\tan^2\omega, b, \theta)$ to calculate, because, when b, ω, θ change to $b^{(n)}, \omega^{(n)}, \theta^{(n)}$, $\overset{2}{\Theta}{}^{(n)}$ vanishes with $b^{(n)}$. In fact it will presently appear that $2^n.\overset{2}{\Theta}{}^{(n)}$ vanishes, when $n = \infty$.

One of the equations marked (37) is

$$\frac{t(\frac{1}{2}\pi - x)}{\frac{1}{2}\pi} + \Theta - \overset{2}{\Omega} = \tan^{-1}\left\{\sin\theta\sin\theta_{0}.\,\frac{b\cos\omega}{\cos\omega^{\circ}}\right\}.$$

Change ω, c, θ, x, t to θ, b, ω, t, x; therefore

$$\frac{x(\frac{1}{2}\pi - t)}{\frac{1}{2}\pi} + \Omega - \overset{2}{\Theta} = \tan^{-1}\{\sqrt{p_1}\sin\omega_1\} = \Psi_1.$$

In the last, change c, ω, θ to c', ω', θ', which changes x to $\frac{1}{2}x$, but leaves t unchanged; therefore

$$\frac{\frac{1}{2}x.(\frac{1}{2}\pi - t)}{\frac{1}{2}\pi} + \Omega' - \overset{2}{\Theta}{}' = \Psi.$$

Eliminate x and t; then

$$(2\Omega' - \Omega) - (2\overset{2}{\Theta}{}' - \overset{2}{\Theta}) = 2\Psi - \Psi_1.$$

But from (42) we have $2\Omega' - \Omega = \Psi$.

Hence $$\overset{2}{\Theta} - 2\overset{2}{\Theta}{}' = (\Psi - \Psi_1) \qquad \text{.................(45)},$$

which is our new equation of reduction.

Repeating it n times, we obtain

$$\overset{2}{\Theta} - 2^n.\overset{2}{\Theta}{}^{(n)} = (\Psi - \Psi_1) + 2(\Psi' - \Psi) + 2^2(\Psi'' - \Psi') + \dots \text{ to } n \text{ terms.}$$

We now need a Lemma, to prove that $2^n.\overset{2}{\Theta}{}^{(n)}$ vanishes when $n = \infty$.

First, observe that when c is so small that c^4 and $c^4\tan^4\theta$ are omissible, and $c^2\tan^2\theta$ is the parameter, we have the following values:

$$\Pi(c^2\tan^2\theta, c, \omega) = \int_0 (1 - c^2\tan^2\theta\sin^2\omega)(1 + \tfrac{1}{2}c^2\sin^2\omega)\,d\omega,$$

$$= \omega - \tfrac{1}{2}c^2(\tan^2\theta - \tfrac{1}{2})(\omega - \tfrac{1}{2}\sin 2\omega),$$

$$\Pi_c(c^2\tan^2\theta) = \tfrac{1}{2}\pi\{1 - \tfrac{1}{2}c^2(\tan^2\theta - \tfrac{1}{2})\},$$

$$F(c\omega) = \omega + \tfrac{1}{4}c^2(\omega - \tfrac{1}{4}\sin 2\omega),$$

$$F_c = \tfrac{1}{2}\pi(1 + \tfrac{1}{4}c^2):$$

and combining these, we get

$$\mathrm{P} = \tfrac{1}{4}c^2\tan^2\theta\sin 2\omega = \tfrac{1}{4}p\sin 2\omega.$$

Also $$\sqrt{T} = \frac{\Delta(b\theta)}{\sin\theta\cos\theta},$$

which converges to $\dfrac{1}{\sin\theta}$, therefore

$$\sqrt{T}\mathrm{P} = \tfrac{1}{4}\,\frac{c^2\tan^2\theta}{\sin\theta}\,.\sin 2\omega.$$

Similarly then $\overset{2}{\Theta}$, when b is very small, converges to

$$\tfrac{1}{4}\cdot\frac{b^2\tan^2\omega}{\sin\omega}\cdot\sin 2\theta:$$

and when we change b, θ, ω into $b^{(n)}, \theta^{(n)}, \omega^{(n)}$, we know that $\theta^{(n)}, \omega^{(n)}$ approach to fixed limits t and $\grave{\omega}$ (which are less than $\tfrac{1}{2}\pi$, if θ, ω are less), so that we have nearly

$$\overset{2}{\Theta}{}^{(n)} = \tfrac{1}{4}b^{(n)2}.\frac{\tan^2\grave{\omega}}{\sin\grave{\omega}}.\sin 2t = \mathrm{k}.b^{(n)2};$$

the quantity k being finite and independent of n. Hence

$$2^n.\overset{2}{\Theta}{}^{(n)} = \mathrm{k}.\{2^n.b^{(n)2}\}.$$

But when $n = \infty$, $2^n.b^{(n)2}$ is evanescent; and indeed the quantity is extremely small when $n = 3$ or even $n = 2$, if c is very near to 1. Hence we get

$$\overset{2}{\Theta} = (\Psi - \Psi_1) + 2(\Psi' - \Psi) + 2^2(\Psi'' - \Psi') + \&\text{c.}\ldots(46),$$

a series rapidly converging.

When $\overset{2}{\Theta}$ is known, we find Ω or $\overset{2}{\Omega}$ by one of the commutative equations. Thus

$$\Omega = \overset{2}{\Theta} + \Psi_1 - \frac{x}{\frac{1}{2}\pi}(\tfrac{1}{2}\pi - t)$$

$$= \left\{\Psi - x\left(1 - \frac{t}{\frac{1}{2}\pi}\right)\right\} + 2(\Psi' - \Psi) + 2^2(\Psi'' - \Psi') + \&\text{c.}\ldots$$

Θ has necessarily a form of development similar to (46); for as the parameter of Θ is $\cot^2\omega$, which $= b^2\tan^2\omega^\circ$, we need only proceed as if ω°, not ω, had been the original amplitude in Ω, and the result of (46) will be Θ instead of $\overset{2}{\Theta}$. To change ω into ω° does not alter $\sin\omega_1$, nor therefore Ψ_1; but it alters $\Psi, \Psi', \Psi''\ldots$, say into $\Psi^\circ, \Psi'^\circ, \Psi''^\circ\ldots$, therefore

$$\Theta = (\Psi^\circ - \Psi_1) + 2(\Psi'^\circ - \Psi^\circ) + 2^2(\Psi''^\circ - \Psi'^\circ) + \&\text{c.}\ldots(46a).$$

Whether this series or the preceding converges better, seems to depend on the evanescence of $2^n\overset{2}{\Theta}{}^{(n)}$ and $2^n\Theta^{(n)}$; *i.e.* on the magnitude of $\dfrac{\tan^2\omega}{\sin\omega}$, which evidently increases with ω. Hence, if $\omega > \omega^\circ$, (46) seems not to converge so well as (46a); but it converges better, if $\omega < \omega^\circ$. Nevertheless, both converge well, when c is near to 1.

It may farther be observed, that since $\Omega - \Omega' = \Omega' - \Psi$, we have also $\Omega^{(n)} - \Omega^{(n+1)} = \Omega^{(n+1)} - \Psi^{(n)}$. Also since $c, c', c'', c'''\ldots$ $\omega, \omega', \omega'', \omega'''\ldots \theta, \theta', \theta''\ldots$ all tend to fixed limits $1, \grave{\omega}, t$, so do $\Omega', \Omega'', \Omega'''\ldots. \Psi', \Psi'', \Psi'''\ldots$ tend to fixed limits ${}^{`}\Omega\,{}^{`}\Psi$; and since $\Omega^{(n)} - \Omega^{(n+1)}$ has limit zero, so has $\Omega^{(n+1)} - \Psi^{(n)}$; *i.e.* ${}^{`}\Omega = {}^{`}\Psi$;

or $$\sqrt{}\mathrm{TP}(p^{`}, 1, \grave{\omega}) = \tan^{-1}(\sqrt{p}\,\sin\grave{\omega});$$

or, since $p^{`}\grave{\omega}$ are mutually independent, we have generally

$$\sqrt{}\mathrm{TP}(p, 1, \omega) = \tan^{-1}(\sqrt{p}\,\sin\omega) \ldots\ldots\ldots\ldots (47),$$

which may be easily confirmed by direct integration.

Thus in equation (42), as also in the reciprocal, and in the conjugate equation, the function *tan*$^{-1}$ may be replaced by an integral of the form $\sqrt{}\mathrm{TP}(\cot^2\mu, 1, \psi)$. In equation (8) we might similarly substitute

$$CG(c\omega) - C_1 G(c_1\omega_1) = C_1 c_1 G(1, \omega_1).$$

It is remarkable how the function $\tan^{-1}(\sqrt{p_1}\,\sin\omega_1)$ derived according to Lagrange's and Gauss's scale seems to intrude itself into more elementary equations, as (21), (24).

Finally, it will here be remarked that equation (46) is only in appearance an equation of *ascending* moduli; for though $c, c', c''\ldots$ascend, yet b is the modulus of $\dot{\Theta}$; and $b, b', b'', b'''\ldots$ *decrease* by the same law as $c, c_1, c_2\ldots$ Nevertheless, the mode in which $\theta, \theta', \theta''\ldots\, \omega, \omega', \omega''\ldots$ are derived, is that which we understand to belong to ascending moduli.

XIV. A similar treatment would manifestly apply to the logarithmic P; but Legendre's adaptation of Jacobi's great discovery here supersedes equation (42), by resolving P into a simpler integral. Indeed, Legendre's reduction of Π, or rather of $\sqrt{}\mathrm{T}(\Pi - F)$, to the integral $\Upsilon = \int_0 EdF$, will be well exchanged into a reduction of $\sqrt{}\mathrm{TP}$ to the integral $V = \int_0 GdF$. Of course, as $G = E - \dfrac{E_c}{F_c}F$, so $V = \Upsilon - \frac{1}{2}\dfrac{E_c}{F_c}F^2$: and as G is the *fluctuant* to E, and P to Π, so is V to E. The process will then be as follows:

By Euler's integration, if $F\omega + F\eta = F\zeta$,

$$E\omega + E\eta - E\zeta = c^2 \sin\omega\,\sin\eta\,\sin\zeta,$$

and $$\sin\zeta = \frac{\sin\omega\,\cos\eta\Delta\eta + \sin\eta\,\cos\omega\Delta\omega}{1 - c^2\sin^2\omega\,\sin^2\eta}.$$

When η becomes $-\eta$, let ζ become ε. Observe that we equally have

$$G\omega + G\eta - G\zeta = c^2\sin\omega\,\sin\eta\,\sin\zeta,$$

$$\therefore\; G\zeta - G\varepsilon = 2G\eta - c^2\sin\omega\,\sin\eta\,(\sin\zeta + \sin\eta).$$

Let η be constant; therefore

$$dF\zeta = dF\omega = dF\varepsilon.$$

Hence

$$G\zeta dF\zeta - G\varepsilon dF\varepsilon = 2G\eta dF\omega - c^2 \sin\omega \sin\eta \,.\, \frac{2\sin\omega\cos\eta\Delta\eta}{1 - c^2\sin^2\eta\sin^2\omega}:$$

$$\text{or}\quad \tfrac{1}{2}V\zeta - \tfrac{1}{2}V\varepsilon = G\eta.F\omega - \frac{\Delta\eta}{\tan\eta}.\int_0 \frac{c^2\sin^2\eta\sin^2\omega}{1 - c^2\sin^2\eta\sin^2\omega}\,dF\omega.$$

But the last term

$$= \sqrt{-\mathrm{T}}\,\{\Pi(-c^2\sin^2\eta,\, c,\, \omega) - F\},$$

and, by equation (30),

$$G\eta = \sqrt{-\mathrm{T}}\left\{\frac{\Pi_c(-c^2\sin^2\eta)}{F_c} - 1\right\};$$

which indeed might be here at once inferred by making $\omega = \frac{1}{2}\pi$. Hence

$$\sqrt{-\mathrm{T}}\mathrm{P}(-c^2\sin^2\eta,\, c,\, \omega) = \tfrac{1}{2}V(c\,\varepsilon) - \tfrac{1}{2}V(c\,\zeta)\ldots(48),$$

which throws a new light on equation (34).

The simpler integral $V(c\,\omega)$ now claims a full examination.

XV. Since

$$G(n\pi + \omega) = G\omega, \text{ and } F(n\pi + \omega) = F(n\pi) + F\omega,$$

therefore

$$V(n\pi + \omega) = \int G(n\pi + \omega).dF(n\pi + \omega) = \int G\omega.dF\omega = V(n\pi) + V\omega.$$

Also, since F, E, G are *odd* functions of ω, Υ and V are *even* functions; therefore

$$V(n\pi \pm \omega) = V(n\pi) + V\omega.$$

Let $n = 1$, therefore

$$V(\pi - \omega) = V(\pi + \omega) = V\pi + V\omega.$$

If then $\omega = \pi$, we get

$$0 = V(2\pi) = 2V\pi;\ \text{or}\ V\pi = 0\ldots\ldots\ldots\ldots(49).$$

And since generally $V\{(n+1)\pi\} = V(n\pi) + V\pi$, we prove in succession that $V(2\pi) = 0$, $V(3\pi) = 0$, &c., and generally $V(n\pi) = 0$, when n is integer. Hence

$$V(n\pi \pm \omega) = V\omega\ \ldots\ldots\ldots\ldots\ldots\ldots(50).$$

This property suits V for tabulation, much better than Υ.

If ω begins from 0, and increases to π, V increases while G is positive; that is, up to $\omega = \frac{1}{2}\pi$, where G becomes $G_c = 0$. After this G becomes negative; indeed $G(\pi - \omega) = -G\omega$; so that V decreases after $\omega = \frac{1}{2}\pi$. Consequently, the maximum value of V is at $\omega = \frac{1}{2}\pi$, or when $V = V_c$.

Again, since

$$G\omega + G\omega^\circ = c^2 \sin\omega \sin\omega^\circ = \frac{c^2 \sin\omega \cos\omega}{\Delta\omega},$$

and $$dF\omega = - dF\omega^\circ = \frac{d\omega}{\Delta\omega};$$

multiply these together; therefore

$$dV\omega - dV\omega^\circ = \frac{c^2 \sin\omega \cos\omega d\omega}{1 - c^2 \sin^2\omega};$$

or $$V\omega - V\omega^\circ = \text{const.} - \tfrac{1}{2}\log(1 - c^2 \sin^2\omega).$$

Let $\omega = 0$, $\omega^\circ = \frac{1}{2}\pi$; therefore const. $= - V_c$, or

$$V\omega^\circ - V\omega = V_c + \log\Delta\omega \text{ .... } (51).$$

Cor. Let $\omega = \frac{1}{2}\pi$, $\omega^\circ = 0$; therefore

$$- 2V_c = \log b, \quad V_c = \tfrac{1}{2}\log b^{-1} \text{ .... } (51a).$$

Thus, if b be infinitesimal, V_c is infinite. Nevertheless, even for small values of b, V_c is of very moderate amount, since it is only a logarithm.

When c is infinitesimal, $G(c\,\omega)$ vanishes for all values of ω; hence so also does $V(c\,\omega)$.

So if, $F\psi = 2F\omega$, or $\sin\psi = \dfrac{2\sin\omega\cos\omega\Delta\omega}{1 - c^2\sin^4\omega}$,

and $G\psi = 2G\omega - c^2\sin^2\omega\sin\psi$, we get

$$V\psi = \textstyle\int_0 G\psi.dF\psi = \int_0 \{2G\omega - c^2\sin^2\omega\sin\psi\}\, 2dF\omega,$$

or $$V\psi - 4V\omega = \int_0 - 2c^2\sin^2\omega \,.\, \frac{2\sin\omega\cos\omega\Delta\omega}{1 - c^2\sin^4\omega} \,.\, \frac{d\omega}{\Delta\omega}$$

$$= \log(1 - c^2\sin^4\omega) \text{ .... } (52).$$

Equations (51), (51a), (52) are in close analogy with those established by Legendre concerning the function Υ, the integrations being the very same.

XVI. We proceed to apply Lagrange's scale to V.

Since $CG = C_1G_1 + C_1c_1\sin\omega_1$,

or $$CG - C_1G_1 = \tfrac{1}{2}Cc^2\frac{\sin\omega\cos\omega}{\Delta(c\omega)},$$

also $$\frac{dF}{C} = \tfrac{1}{2}\frac{dF_1}{C_1} = \frac{d\omega}{C\Delta(c\omega)};$$

multiply the two equations; therefore

$$dV - \tfrac{1}{2}dV_1 = \tfrac{1}{2}c^2\frac{\sin\omega\cos\omega d\omega}{1 - c^2\sin^2\omega};$$

or $$V - \tfrac{1}{2}V_1 = - \tfrac{1}{2}\log\Delta(c\omega) \text{ .... } (53),$$

which, for a logarithmic Π, replaces (42) for the circular Π.

As we have $C\{G(c\omega) - G(c\omega^\circ)\} = 2C_1 G(c_1\omega_1)$
so also $V(c\omega) + V(c\omega^\circ) = V(c_1\omega_1) + V_c$ ......(54).

If we repeat (53) n times, we find

$$V - 2^{-n} V_n = -2^{-1}\log\Delta - 2^{-2}\log\Delta_1 - 2^{-3}\log\Delta_2 - \text{\&c. to } n \text{ terms.}$$

Also, since $c_n = 0$, when $n = \infty$, so is $V_n = 0$; therefore

$$V = -\tfrac{1}{2}\log\Delta - \tfrac{1}{4}\log\Delta_1 - \tfrac{1}{8}\log\Delta_2 - \text{\&c.} \;\ldots\ldots(55),$$

which is analogous to the development (44). In fact, applying the last to (48), and observing that to form $\eta_1\eta_2\eta_3\ldots$ from η, and $\omega_1, \omega_2, \omega_3\ldots$ from ω, by Lagrange's scale, and then to form $\zeta, \zeta_1, \zeta_2, \zeta_3\ldots$ by coupling ω and η, ω_1 and η_1, &c. amounts to forming $\zeta, \zeta_1, \zeta_2, \zeta_3\ldots$ by Lagrange's scale; and similarly of $\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3\ldots$, we get

$$\sqrt{-}TP(-c^2\sin^2\eta, c, \omega) = \tfrac{1}{2}V(c\varepsilon) - \tfrac{1}{2}V(c\zeta)$$

$$= \tfrac{1}{4}\log\frac{\Delta\zeta}{\Delta\varepsilon} + \tfrac{1}{8}\log\frac{\Delta\zeta_1}{\Delta\varepsilon_1} + \tfrac{1}{16}\log\frac{\Delta\zeta_2}{\Delta\varepsilon_2} + \text{\&c.}$$

$$= -\tfrac{1}{4}\log\frac{1+D_1}{1-D_1} - \tfrac{1}{8}\log\frac{1+D_2}{1-D_2} - \tfrac{1}{16}\log\frac{1+D_3}{1-D_3} - \text{\&c.}\ldots(56),$$

if $D = c\sin\eta\sin\omega$. This equation is the transformation of (44) to the case of a logarithmic P, and apparently must be actually used to approximate to $\sqrt{-}TP$, until tables of V are calculated.

When c is very near to 1, we may invert the method. Give to (53) the form

$$V = \log\Delta' + 2V',$$

or $$(V + \log\Delta) = (\log\Delta - \log\Delta') + 2(V' + \log\Delta').$$

Repeating this n times, we get

$$V + \log\Delta = \log\frac{\Delta}{\Delta'} + 2\log\frac{\Delta'}{\Delta''} + \ldots + 2^{n-1}\log\frac{\Delta^{(n-1)}}{\Delta^{(n)}} + 2^n(V^{(n)} + \log\Delta^{(n)}).$$

Now $2^n G^{(n)} = 2^n E^{(n)} - 2^n.\dfrac{F^{(n)}}{C^{(n)}}.E_c^{(n)}$. Also $\dfrac{F}{C} = 2^n.\dfrac{F^{(n)}}{C^{(n)}}$,

$$\therefore\; 2^n G^{(n)} = 2^n.E^{(n)} - \frac{F}{C}.E_c^{(n)}.$$

Again, when b is very small,

$$E = \sin\omega + \tfrac{1}{2}b^2\textstyle\int_0 \tan^2\omega\, d\sin\omega,$$

whence $2^n.E^{(n)} = 2^n.\sin\omega^{(n)} + 2^{n-1}b^{(n)2}.\int_0\tan^2\omega^{(n)}d\sin\omega^{(n)}$.

Now if $\omega < \frac{1}{2}\pi$, the series $\omega, \omega', \omega'', \omega'''$... decrease towards a limit $\grave{\omega} < \frac{1}{2}\pi$, so that the last integral is finite and independent of n; while $2^n b^{(n)2}$ is infinitesimal when n is infinite. Hence, for infinite values of n, $2^n E^{(n)} = 2^n. \sin\grave{\omega}$.

It is still easier to see that $\Delta^{(n)} = \cos\grave{\omega}$, when $n = \infty$.

Also $E_c^{(n)} = 1$. Hence

$$2^n.G^{(n)} = 2^n. \sin\grave{\omega} - \frac{F}{C}.$$

But $\dfrac{F^{(n)}}{B^{(n)}} = \dfrac{F}{B}$, and $B^{(n)}$ converges to $\frac{1}{2}\pi$, $F^{(n)}$ to $\dfrac{d\grave{\omega}}{\cos\grave{\omega}}$.

$$\therefore\ 2^n.V^{(n)} \text{ or } 2^n.\int_0 G^{(n)}dF^{(n)} = 2^n.\int_0 \frac{\sin\grave{\omega}}{\cos\grave{\omega}}\,d\grave{\omega} - \tfrac{1}{2}\pi\int_0 \frac{FdF}{BC},$$

$$\left.\begin{aligned} &2^n\{V^{(n)} + \log\cos\grave{\omega}\} = -\tfrac{1}{4}\pi.\frac{F^2}{BC} \\ \text{or}\quad &2^n\{V^{(n)} + \log\Delta^{(n)}\} = -\tfrac{1}{4}\pi.\frac{F^2}{BC}\end{aligned}\right\}\text{ when } n = \infty \ldots(57).$$

Finally, then,

$$V = -\tfrac{1}{4}\pi.\frac{F^2}{BC} + \log\frac{1}{\Delta'} + 2\log\frac{\Delta'}{\Delta''} + 2^2\log\frac{\Delta''}{\Delta'''} + \&c.\ldots(58),$$

which converges excellently when c is near to 1.

XVII. As we had a new integral H so related to G that $H - G = \frac{1}{2}\pi.\dfrac{F}{BC}$, it is well to conceive of W similarly related to V. Namely, as $V = \int_0 GdF$, so let $W = \int_0 HdF$;

$$\therefore\ W - V = \tfrac{1}{4}\pi\frac{F^2}{BC}\ldots\ldots\ldots\ldots(59).$$

This indicates that the last series is a development of W, analogous to (12),

$$W = \log\frac{1}{\Delta'} + 2\log\frac{\Delta'}{\Delta''} + 2^2\log\frac{\Delta''}{\Delta'''} + \&c.\ldots\ldots(60).$$

Again, if $\sin\eta = \sqrt{-1}\tan\theta$, we obtain from $F(c\eta) = \sqrt{-1}F(b\theta)$, and from (13),

$$\left.\begin{aligned} V(c\eta) &= W(b\theta) + \log\cos\theta \\ W(c\eta) &= V(b\theta) + \log\cos\theta\end{aligned}\right\}\ldots\ldots\ldots\ldots(61),$$

in which it is remarkable that $\sqrt{-1}$ has vanished entirely. In truth, the transformation $\sin\eta = \sqrt{-1}\tan\theta$ is nothing but

a device for enabling the sines and cosines of Trigonometry to do duty for *hyperbolic* sines and cosines; and we might in all cases evade $\sqrt{-1}$ in this transformation by having recourse to hyperbolic functions. If capital letters denote these, thus,

$$\mathrm{Cos}\,x = \tfrac{1}{2}(e^x + e^{-x}),\ \mathrm{Sin}\,x = \tfrac{1}{2}(e^x - e^{-x}),\ \&c.,$$

so that $\mathrm{Cos}^2 x - \mathrm{Sin}^2 x = 1$, and $1 - \mathrm{Tan}^2 x = \mathrm{Sec}^2 x$;

it is manifest that, by assuming $\sin\omega = \mathrm{Tan}\,x$, we get

$$\cos\omega = \mathrm{Sec}\,x,\ \sec\omega = \mathrm{Cos}\,x,\ \tan\omega = \mathrm{Sin}\,x,$$
$$d\omega = \mathrm{Sec}\,x dx,\ dx = \sec\omega d\omega.$$

Hence $F(c\omega) = \int_0 \frac{\sec\omega d\omega}{\sqrt{(1+b^2\tan^2\omega)}} = \int_0 \frac{dx}{\sqrt{(1+b^2\,\mathrm{Sin}^2 x)}} = \phi(b, x).$

Thus, by adopting the double form F and ϕ, we might deal with *real* functions only; and equations (61) seem to indicate that the integrals of the second order, V and W, recover common sines and cosines, which were displaced by hyperbolic sines and cosines in the integrals of the first order.

The equations (61) facilitate many transformations.

We may moreover give another form to (58) by slightly modifying the equation of reduction.

Since $V = \log\Delta' + 2V'$, $V + \log\cos\omega = \log(\Delta'\cos\omega) + 2V'$ $= \log(\Delta'\cos\omega\sec^2\omega') + 2(V' + \log\cos\omega')$. But $\Delta'.\cos\omega\sec^2\omega'$ $= 1 - b'\tan^2\omega'$; therefore

$$V + \log\cos\omega = \log(1 - b'\tan^2\omega') + 2(V' + \log\cos\omega')\ldots(62).$$

Repeat this n times; observe that $2^n(V^{(n)} + \log\cos\omega^{(n)})$ converges to $-\tfrac{1}{4}\pi.\frac{F^2}{BC}$; therefore

$$W + \log\cos\omega = \log(1 - b'\tan^2\omega') + 2\log(1 - b''\tan^2\omega'')$$
$$+ 2^2\log(1 - b'''\tan^2\omega''') + \&c.$$

Change $\sin\omega$ into $\sqrt{-1}\tan\theta$, $\sin\omega'$ into $\sqrt{-1}\tan\theta'$, &c. Observe that, by (61), we get

$$W(c\omega) = V(b\theta) + \log\cos\theta = V(b\theta) - \log\cos\omega;$$

hence $V(b\theta) = \log(1 + b'\sin^2\theta') + 2\log(1 + b''\sin^2\theta'') + \&c.,$

where $\theta, \theta', \theta''\ldots$ follow the scale of Gauss,

$$\sin\theta = \frac{(1+b')\sin\theta'}{1 + b'\sin^2\theta'},\ \text{or}\ 1 + b'\sin^2\theta' = \frac{c'}{\sqrt{c}}.\frac{\sin\theta'}{\sin\theta};$$

or $$V(b\theta) = \log\left(\frac{c'}{\sqrt{c}}.\frac{\sin\theta'}{\sin\theta}\right) + 2\log\left(\frac{c''}{\sqrt{c'}}.\frac{\sin\theta''}{\sin\theta'}\right) + \&c.$$

Let $\theta = \frac{1}{2}\pi$, therefore $\theta' = \frac{1}{2}\pi = \theta'' = \theta''' = \&c....$, therefore

$$V_b = \log\frac{c'}{\sqrt{c}} + 2\log\frac{c''}{\sqrt{c'}} + 2^2\log\frac{c'''}{\sqrt{c''}} + \&c.... = -\log\sqrt{c},$$

and $V(b\theta) = V_b + \log\dfrac{\sin\theta'}{\sin\theta} + 2\log\dfrac{\sin\theta''}{\sin\theta'} + 2^2\log\dfrac{\sin\theta'''}{\sin\theta''} + \&c....(63)$.

This is adapted to the case of b very small, and b is here the modulus: hence the new series has no real advantage; for it is less convenient than (55). Yet we may step back to $W(c\omega)$, and write

$$1 - b'\tan^2\omega' = \frac{c'}{\sqrt{c}}\cdot\frac{\tan\omega'}{\tan\omega};$$

therefore $W(c\omega) + \log\cos\omega$

$$= V_b + \log\frac{\tan\omega'}{\tan\omega} + 2\log\frac{\tan\omega''}{\tan\omega'} + 2^2\log\frac{\tan\omega'''}{\tan\omega''} + \&c....(64),$$

which is adapted to the case of c near to 1.

For $\log\cos\omega - V_b$ we may write $\log(\sqrt{c}\cos\omega)$.

A new development of G by the scale of Gauss, bearing analogy to (63), may deserve notice. Since

$$V(b\theta) = \log(1 + b'\sin^2\theta') + 2V(b'\theta');$$

as the series itself indicates; we get, by differentiating, since

$$\frac{F(b\theta)}{B} = \frac{F(b'\theta')}{B'},$$

$$BG(b\theta) = \frac{2b'\sin\theta'\cos\theta'}{1 + b'\sin^2\theta'}.B'\Delta(b'\theta') + 2B'G(b'\theta')$$

$$= 2B'b'\sin\theta'\cos\theta + 2B'G(b'\theta')\text{.........}(63a).$$

It is easy to shew that $2^n.G(b^{(n)}\theta^{(n)})$ vanishes when $n = \infty$; therefore

$BG(b\theta) = 2B'b'\sin\theta'\cos\theta$

$$+ 2^2B''b''\sin\theta''\cos\theta' + 2^3B'''b'''\sin\theta'''\cos\theta'' + \&c....(63b).$$

But this is less simple than (8).

XVIII. To approximate to $V(c\omega)$ is now as easy as to find $G(c\omega)$ or $E(c\omega)$, except only that we have no tables of it ready calculated. But it is not without interest to consider the result of differentiating V with reference to c as the variable; for which purpose we step back to F, E, and G.

Let $a = \dfrac{B}{C}$, $x = \frac{1}{2}\pi\,.\,\dfrac{F(c\omega)}{C}$. Then, in the scale whose

index is n, to pass from $c\omega$ to new elements $c_1\omega_1$ changes ax to na,* nx. Moreover, in the higher theory, if we adopt, with Dr. Gudermann, the notation of *hyperbolic* sines and cosines, the development of G admits of the form

$$CG(c\omega) = \pi \left\{\frac{\sin 2x}{\operatorname{Sin}\pi a} + \frac{\sin 4x}{\operatorname{Sin}2\pi a} + \frac{\sin 6x}{\operatorname{Sin}3\pi a} + \&c....\right\},$$

which things suggest the advantage of making a rather than c the base of variation.

The same process which demonstrates $\frac{1}{2}\pi = F_cE_b + F_bE_c - F_bF_c$ shews, in passing, that $\frac{1}{2}\pi.\dfrac{dc}{da} = -b^2c.F_c^2$. In the common treatises we have

$$\frac{dF}{dc} = \frac{E}{b^2c} - \frac{F}{c} - \frac{c\sin\omega\cos\omega}{b^2\Delta\omega}, \text{ when } \omega \text{ is constant;}$$

$$\therefore\ \tfrac{1}{2}\pi.\frac{dF}{da} = -F_c^2\{E - b^2F - c^2\sin\omega\sin\omega^\circ\}.....(65).$$

Hence $\quad \frac{1}{2}\pi.\dfrac{dF_c}{da} = -F_c^2\{E_c - b^2F_c\}$, when $\omega = \frac{1}{2}\pi$.

Multiply the last by $\dfrac{F}{F_c}$ and subtract from (65), therefore

$$\tfrac{1}{2}\pi\left\{\frac{dF}{da} - \frac{F}{F_c}.\frac{dF_c}{da}\right\} = -F_c^2\{G - c^2\sin\omega\sin\omega^\circ\},$$

whence $\frac{1}{2}\pi.\dfrac{d}{da}\left(\dfrac{F}{F_c}\right)$ or $\dfrac{dx}{da} = F_c.G(c\omega^\circ)$, when ω is const...(66).

Let A stand temporarily for CG or F_cG; *i.e.* $A = F_cE - E_cF$; and let us seek for $\dfrac{dA}{da}$.

$$\text{Since } \frac{dE}{dc} = -\frac{F-E}{c},\quad \tfrac{1}{2}\pi.\frac{dE}{da} = b^2F_c^2(F-E).$$

$$\text{Also } \tfrac{1}{2}\pi.\frac{dA}{da} = F_c.\tfrac{1}{2}\pi\frac{dE}{da} - F.\tfrac{1}{2}\pi.\frac{dE_c}{da} + E.\tfrac{1}{2}\pi\frac{dF_c}{da} - E_c.\tfrac{1}{2}\pi.\frac{dF}{da}$$

$$= -E_cF_c^2c^2\sin\omega\sin\omega^\circ \text{ ....(67)},$$

by mere substitutions.

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* For $a = \dfrac{B}{C}$, $a_1 = \dfrac{B_1}{C_1}$ gives $a = \dfrac{1}{n}a_1$ in that scale.

It is observable, that we also have

$$\tfrac{1}{2}\pi.\frac{d(cF_c)}{da} = -E_cF_c^2c\,;$$

so that
$$\frac{dA}{da} = \sin\omega\sin\omega^\circ.\frac{d(cF_c)}{da} \;\ldots\ldots\ldots\ldots\; (67a).$$

In all these equations, (65)—(67a), we suppose ω to be constant. But in general, when c and ω both vary,

$$\frac{d(x)}{da} = \frac{dx}{d\omega}\frac{d\omega}{da} + \frac{dx}{da}\,;\;\text{ and }\;\frac{d(A)}{da} = \frac{dA}{d\omega}\frac{d\omega}{da} + \frac{dA}{da}\,;$$

that is,
$$\frac{d(x)}{da} = \frac{\frac{1}{2}\pi}{F_c\Delta}\cdot\frac{d\omega}{da} + A(\omega^\circ)\,;$$

$$\tfrac{1}{2}\pi\frac{d(A)}{da} = \tfrac{1}{2}\pi\frac{d\omega}{da}\cdot\left(\frac{dA}{d\omega}\right) - E_cF_c^2c^2\sin\omega\sin\omega^\circ.$$

Now let x be the principal variable instead of ω; and when a varies, let x be constant, or $\dfrac{d(x)}{da} = 0$; and eliminate $\dfrac{d\omega}{da}$ from the two last; observing that

$$\frac{dA}{d\omega} = F_c\Delta - \frac{E_c}{\Delta}\;\text{ and }\;A(\omega^\circ) = F_cc^2\sin\omega\sin\omega^\circ - A(\omega)\,;$$

$$\therefore\;\tfrac{1}{2}\pi.\frac{dA}{da} = F_c.\Delta.A\frac{dA}{d\omega} - F_c^3c^2\sin\omega\cos\omega\Delta \;\ldots\ldots(68).$$

Multiply by $\dfrac{dx}{\frac{1}{2}\pi} = \dfrac{d\omega}{F_c\Delta}$; and since x is constant in $\dfrac{d}{da}$, we may integrate for x under the d; or

$$\int_0\frac{dA}{da}\,dx = \frac{d}{da}\int_0 Adx = \frac{d}{da}\int_0 GdF.\tfrac{1}{2}\pi\,;$$

$$\therefore\;\tfrac{1}{2}\pi.\frac{d}{da}V(c\omega) = \tfrac{1}{2}A^2(c\omega) - \tfrac{1}{2}F_c^2c^2\sin^2\omega \;\ldots\ldots\; (69),$$

when x is constant.

If this have no other interest, it at least shews that A^2 can be expressed in series of the cosines of $2x$ and of its multiples: for the developments of $V(c\omega)$ and $F_c^2c^2\sin^2\omega$ are known.

XIX. The same notation facilitates the management of E, G, and V in the higher scales. To avoid confusion, in the scale whose index is n, let $h\psi$ be the new elements of F

which are called $c_1\omega_1$ in the common scale. Then, since x, a change to nx, na, when $c\omega$ change to $h\psi$, we have, as *total* variations,

$$\left.\begin{aligned}\frac{d.x}{da} &= \frac{\frac{1}{2}\pi}{F_c\Delta(c\omega)}\cdot\frac{d\omega}{da} + F_cG(c\omega^\circ)\\ \frac{d.nx}{nda} &= \frac{\frac{1}{2}\pi}{F_h\Delta(h\psi)}\cdot\frac{d\psi}{nda} + F_hG(h\psi^\circ)\end{aligned}\right\}.$$

The left-hand member is the same in both equations, and we may at pleasure assume any *one* of the variables as constant. If it be ω, $\frac{d\omega}{da} = 0$, therefore

$$F_cG(c\omega^\circ) = F_hG(h\psi^\circ) + \frac{\frac{1}{2}\pi}{F_h\Delta(h\psi)}\cdot\frac{d\psi}{nda}\ \ldots\ldots(70).$$

This is a generalization of $CG = C_1G_1 + C_1c_1\sin\omega_1$, and supersedes a much more complicated one connecting $E(c\omega)$ with $E(h\psi)$ in Legendre.

Multiply by

$$\frac{dF(c\omega^\circ)}{F_c} = \frac{dF(h\psi^\circ)}{nF_h} = -\frac{1}{nF_h}\cdot\frac{d\psi}{\Delta(h\psi)};$$

$$\therefore\ V(c\omega^\circ) = \frac{1}{n}V(h\psi^\circ) - \frac{\frac{1}{2}\pi}{n^2F_h^2}\cdot\int_0\left(\frac{d\psi}{da}\right)\frac{d\psi}{1-h^2\sin^2\psi}\ \ldots(71),$$

which also is a generalization of (53).

Since Jacobi's equations shew $\frac{d\psi}{da}$ to be a rational trigonometrical function, the integral is of a lower order than F.

In Legendre's own scale, the index of which is 3, equation (70) takes the form

$$F_cG(c\omega) = F_hG(h\psi) + \tfrac{2}{3}F_c.\mu c^2\sin^2\alpha_2.\ \frac{\sin\omega\cos\omega\Delta(c\omega)}{1-c^2\sin^2\alpha_2\sin^2\omega}\ \ldots(70a),$$

where $F(c\alpha_2) = \frac{2}{3}F_c$, and $\tan\frac{1}{2}(\psi-\omega) = \frac{1-\mu}{\mu}\tan\omega$.

Multiply by $\frac{dF(c\omega)}{F_c} = \frac{dF(h\psi)}{3F_h}$, and in place of (71) we have

$$V(c\omega) = \tfrac{1}{3}V(h\psi) - \tfrac{1}{3}\mu\log(1-c^2\sin^2\alpha_2\sin^2\omega)\ldots(71a).$$

This easily gives a new development of V, converging far more rapidly than (53): but until we have tables for the trisection of F_c, the trouble of calculating the constants makes this scale practically useless.

It is proper also to notice here the relation borne by V and W to Jacobi's new functions. Let q be a small fraction such that $\log q^{-1} = \pi a$, and $\Theta\Lambda$ functions such that

$$\left.\begin{array}{l}\Theta = 1 - 2q\cos 2x + 2q^{2.2}\cos 4x - 2q^{3.3}\cos 6x + \&c.... \\ \Lambda = \quad 2q^{\frac{1}{4}}\sin x - 2q^{\frac{3.3}{4}}\sin 3x + 2q^{\frac{5.5}{4}}\sin 5x - \&c....\end{array}\right\};$$

then, among other equations, Jacobi has proved that

$$\sqrt{(1 - c^2\sin^2\omega)} = \sqrt{b}.\frac{\Theta(q, \frac{1}{2}\pi + x)}{\Theta(qx)},$$

$$\text{and } \cos\omega = \sqrt{\frac{b}{c}}.\frac{\Lambda(q, x + \frac{1}{2}\pi)}{\Theta(qx)}.$$

Legendre has demonstrated (2nd Supplement, § XI.) that

$$\frac{2F_c}{\pi}G(c\omega) = \frac{\Theta'(qx)}{\Theta(qx)}.$$

If we multiply by $\frac{1}{2}\pi.\dfrac{dF(c\omega)}{F_c} = dx$, and integrate, we get

$$V = \log(\beta\Theta),$$

where β is a function of c.

When $\omega = \pi$, $V = 0$; therefore $\beta\Theta = 1$, or

$$\beta^{-1} = \Theta(q, \pi) = 1 - 2q + 2q^{2.2} - 2q^{3.3} + \&c....,$$

a series which is known to be equal to $\sqrt{\dfrac{2bF_c}{\pi}}$.

But it suffices to write $V = \log\dfrac{\Theta(q, x)}{\Theta(q, \pi)}$ ....(72).

This elegant relation is obscurely expressed by Legendre under the following form

$$\log\Theta(qx) = \Upsilon(c\omega) - \frac{1}{2}\frac{E_c}{F_c}.F^2(c\omega) + \frac{1}{2}\log\frac{2bF_c}{\pi}.$$

Had he used the integral V, he would not have overlooked the following curious inference, which would certainly have had a charm for him.

Since (by the form of Θ when resolved into factors)

$$\Theta(q^n, nx) = \text{const.} \times \Theta(q, x).\Theta\left(q, x + \frac{\pi}{n}\right).\Theta\left(q, x + \frac{2\pi}{n}\right)......$$
$$\Theta\left(q, x + \frac{n-1}{n}\pi\right):$$

differentiate logarithmically, observing that

$$d \log \Theta = dV = GdF,$$

and putting h the same function of nx as c is of x. Therefore

$$nF_h . G(q^n, nx)$$

$$= F_c \left\{ G(q, x) + G\left(q, x + \frac{\pi}{n}\right) + \ldots + G\left(q, x + \frac{n-1}{n}\pi\right) \right\} \ldots (73),$$

in which we write after G the elements qx instead of $c\omega$, but meaning the same quantity.

A process which is laborious in Legendre becomes easy by aid of (72), (59), (61).

When $\sin\omega = \sqrt{-1} \tan\theta$, and $F(c\omega) = \sqrt{-1}.F(b\theta)$,

$$x = \tfrac{1}{2}\pi . \frac{F(c\omega)}{C} = \sqrt{-1} . \frac{B}{C} . \frac{F(b\theta)}{B} = \sqrt{-1}.at.$$

Call $T(qt)$ the value assumed by $\Theta(q, x)$; that is,

$$\Theta(qx) = 1 - 2q \operatorname{Cos} 2at + 2q^{2.2} \operatorname{Cos} 4at - \&c. \ldots = T(q, t),$$

$$\therefore \log \beta T(qt) = V(c\omega) = W(b\theta) + \log\cos\theta, \text{ by (61)},$$

$$= V(b\theta) + \tfrac{1}{4}\pi . \frac{F^2(b\theta)}{BC} + \log\cos\theta, \text{ by (59)},$$

$$= \log \gamma\Theta(rt) + \tfrac{1}{4}\pi a t^2 + \log\cos\theta,$$

if q, β become r, γ when c changes to b. Giving to β, γ their values, we get

$$\log T(qt) = \log\Theta(rt) + \tfrac{1}{4}\pi a t^2 + \log\cos\theta + \tfrac{1}{2}\log\frac{b}{ca} \ldots (74),$$

which is one of Legendre's equations, and, by means of $\cos\omega = \sqrt{\frac{b}{c}} . \frac{\Lambda(q, x + \frac{1}{2}\pi)}{\Theta(qx)}$, readily gives his beautiful result

$$\left.\begin{aligned} \sqrt{a}.T(qt) &= q^{-\left(\frac{t}{\pi}\right)^2} . \Lambda(r, t + \tfrac{1}{2}\pi) \\ \sqrt{a^{-1}}.T(rx) &= r^{-\left(\frac{x}{\pi}\right)^2} . \Lambda(q, x + \tfrac{1}{2}\pi) \end{aligned}\right\} \ldots\ldots\ldots (75);$$

so that Λ is found from T, or T from Λ, according as one or the other converges best.

In this part of the subject it must be added, that the arc Ω, elicited by Legendre from Jacobi's theory, is no other quantity than that which I have called $\sqrt{TP}$; so that,

according to the meaning of $\overset{2}{\Omega}$ in equation (26),

$$\tan\overset{2}{\Omega} = \frac{2q\sin 2x.\mathrm{Sin}\,2at - 2q^{2.2}\sin 4x.\mathrm{Sin}\,4at + 2q^{3.3}\sin 6x\,\mathrm{Sin}\,6at - \&c.}{1 - 2q\cos 2x.\mathrm{Cos}\,2at + 2q^{2.2}\cos 4x.\mathrm{Cos}\,4at - \&c.\dots}.$$

But it suffices to mention the fact. Dr. Gudermann has several other elegant approximations to $\overset{2}{\Omega}$ by this higher theory.

I have thought that we might call the function $\sqrt{\pm}\,T\{\Pi - F\}$ the *Principal Compound* of the Third Elliptic Species, and $\sqrt{\pm}\,TP$ its *Fluctuant:* also G the Fluctuant of E, H its Companion; Υ the First Integral of the Second Order, V its Fluctuant, and W the Companion of V. To avoid a perpetual appropriation of capital letters, some such notation as *flE* for G, $fl\Pi$ for P, &c. may sometimes be advisable.

XX. Lastly, it may be worth while to touch on a neglected side of the subject; but, as it involves no difficulty of principle, and possibly is more curious than useful, I may be brief.

In fixing the standard forms of F, E, Π, two arbitrary limitations are introduced,—to use circular sines, and not hyperbolic; and, to make the multiplier a negative proper fraction $(-c^2)$. Imaginary Amplitudes overthrow the former limitation, imaginary Moduli the other. But to change $-c^2$ into $+c^2$ involves nothing imaginary, nor indeed, within certain limits, to change b^2 into b^{-2}, which makes $c^2 > 1$.

For a moment put $z = \tan\omega$, and let $\psi(bz)$, $\chi(bz)$ denote what $F(c\omega)$, $E(c\omega)$ become; or

$$\psi(bz) = \int_0 \frac{dz}{\sqrt{(1+z^2)}\sqrt{(1+b^2z^2)}}, \quad \chi(bz) = \int_0 \sqrt{\left(\frac{1+b^2z^2}{1+z^2}\right)}\cdot\frac{dz}{1+z^2}.$$

Mere inspection of these shews that

$$d\psi(b^{-1}, z^{-1}) = -\,b.\psi(bz) \text{ and } d\chi(b^{-1}, z^{-1}) = -\,b^{-1}.\chi(bz).$$

To change z into z^{-1} changes ω into $\frac{1}{2}\pi - \omega$; and if $c = \sin\gamma$, to change b into b^{-1}, or $\cos\gamma$ into $\sec\gamma$, is equivalent to changing $\sin\gamma$ into $\sqrt{-1}\tan\gamma$. Hence, integrating the two last, we find, if $\omega + \theta = \frac{1}{2}\pi$,

$$\left.\begin{aligned} b^{-1}F(\sqrt{-1}\tan\gamma, \theta) + F(\sin\gamma, \omega) &= F_c \\ bE(\sqrt{-1}\tan\gamma, \theta) + E(\sin\gamma, \omega) &= E_c \end{aligned}\right\} \dots\dots(76).$$

It is easy then to transform the developments which express F_cF_b and E_cE_b in terms of c^2, into others in which $-c^2b^{-2}$ stands for $+c^2$. For

$$b^{-1}F(\sqrt{-1}\tan\gamma, \tfrac{1}{2}\pi) = F_c \text{ and } bE(\sqrt{-1}\tan\gamma, \tfrac{1}{2}\pi) = E_c \dots(76a).$$

It farther follows that

$$bG(\sqrt{-1}\tan\gamma,\theta) = b.E(\sqrt{-1}\tan\gamma,\theta) - b^{-1}\frac{E_c}{F_c}F(\sqrt{-1}\tan\gamma,\theta)$$

$$= \{E_c - E(\sin\gamma,\omega)\} - \frac{E_c}{F_c}\{F_c - F(\sin\gamma,\omega)\}$$

$$= - G(\sin\gamma,\omega)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(77).$$

Hence also, if A stands for $F_cE - E_cF$, we obtain

$$A(\sqrt{-1}\tan\gamma,\theta) = - A(\sin\gamma,\omega)\ldots\ldots\ldots(78).$$

Moreover, since $V = \int_0 GdF$,

$$V(\sqrt{-1}\tan\gamma,\theta) = \int - b^{-1}G(c\omega)\times - bdF(c\omega) = \int dV(c\omega)$$

$$= V(c\omega) - V_c\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(79).$$

When $\omega = 0$, $\quad V(\sqrt{-1}\tan\gamma,\tfrac{1}{2}\pi) = - V_c$;

which is evidently correct: for since $V_c = \frac{1}{2}\log b^{-1}$, it does but change the sign, if we commute b with b^{-1}.

For (79) we may thus also write

$$V(\sqrt{-1}\tan\gamma,\theta) = V(\sin\gamma,\omega) + \tfrac{1}{2}\log\cos\gamma\ldots(79^*).$$

It might at first seem possible to reduce the circular Π to integrals V which have imaginary moduli; but these equations shew that such integrals fall back into the common form. Moreover, if in Π the modulus is imaginary, the integral is logarithmic exactly when by its parameter it might have been judged to be circular. To exhibit this may well close our subject.

When c changes to $\sqrt{-1}\tan\gamma$, our ordinary logarithmic parameter $-c^2\sin^2\eta$ becomes $\tan^2\gamma.\sin^2\eta$, which is positive and apparently circular. Observe that if $c = \sin\gamma$,

$$\sqrt{(1+\tan^2\gamma\sin^2\eta)}\cot\eta = b^{-1}\sqrt{(1-c^2\cos^2\eta)}\cot\eta,$$

$$\text{and}\quad \frac{dF(\sqrt{-1}\tan\gamma,\theta)}{1+\tan^2\gamma\sin^2\eta\sin^2\theta} = \frac{-b^3.dF(c\omega)}{b^2+c^2\sin^2\eta\cos^2\omega},$$

the denominator of which $=(1-c^2\cos^2\eta)-c^2\sin^2\eta\sin^2\omega$. Hence if we change η all through into $\frac{1}{2}\pi-\eta$, we have

$$\sqrt{-T}\Pi(\tan^2\gamma\cos^2\eta,\sqrt{-1}\tan\gamma,\theta) = -b^2\frac{\Delta(c\eta)}{\cot\eta}.\int\frac{dF(c\omega)}{\Delta^2(c\eta)-c^2\cos^2\eta\sin^2\omega}$$

$$= \frac{-b^2\tan\eta}{\Delta(c\eta)}.\int\frac{dF(c\omega)}{1-c^2\sin^2\eta^\circ.\sin^2\omega} = -\frac{\Delta(c\eta^\circ)}{\tan\eta^\circ}\Pi(-c^2\sin^2\eta^\circ,c,\omega)+\text{const.}$$

Observe that $\dfrac{\Delta(c\eta^\circ)}{\tan\eta^\circ} = \sqrt{-T}(-c^2\sin^2\eta^\circ)$; then we obtain

$$\sqrt{-T}\Pi(\tan^2\gamma\cos^2\eta, \sqrt{-1}\tan\gamma, \theta) + \sqrt{-T}\Pi(-\sin^2\gamma\sin^2\eta^\circ, \sin\gamma, \omega)$$
$$= \sqrt{-T}\Pi(-\sin^2\gamma\sin^2\eta^\circ, \sin\gamma, \tfrac{1}{2}\pi)$$
$$= \sqrt{-T}\Pi_c(-c^2\sin^2\eta^\circ)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (80),$$

where the integrals are all logarithmic, and $\sin\gamma = c$, connecting η and η°.

To transform them into circulars, let

$$\sin\eta^\circ = \sqrt{-1}.\tan\delta_\circ,\ \cos^2\eta = \frac{b^2\sin^2\eta^\circ}{\Delta^2(c\eta^\circ)},$$

$$\text{or }\ \tan^2\gamma\cos^2\eta = \frac{c^2\sin^2\eta^\circ}{1 - c^2\sin^2\eta^\circ} = \frac{-c^2\tan^2\delta^\circ}{1+c^2\tan^2\delta_\circ} = \frac{-\cot^2\delta}{1+\cot^2\delta} = -\cos^2\delta.$$

Hence $\sqrt{T}\Pi(-\cos^2\delta, \sqrt{-1}\tan\gamma, \theta) + \sqrt{T}\Pi(\cot^2\delta, \sin\gamma, \omega)$

$$= \sqrt{T}\Pi_c(\cot^2\delta)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(81).$$

In all of these, from (76) to (81), we suppose $\omega + \theta = \frac{1}{2}\pi$.

ON TWO NEW METHODS OF DEFINING CURVES OF THE SECOND ORDER, TOGETHER WITH NEW PROPERTIES OF THE SAME DEDUCIBLE THEREFROM.

By Professor Steiner.*

(*Extract from a paper read before the Berlin Academy of Science, March* 1852.)

Section I.

The two following methods of generating the Conic Sections are in a measure analogous to, and indeed embrace, the two known methods of generation by means of the two foci, or one focus and the corresponding directrix. The first method consists in making the sum or difference of the lengths of two tangents from the generating point to two given fixed circles, equal to a given constant, instead of, as before, considering the sum or difference of the distances from the said point to the two foci themselves as given and constant. In the second method here employed, the simple directrix is replaced by any number of given

* Translated by Dr. T. A. Hirst.

right lines, perpendiculars are then let fall from the generating point on each of these lines, and a certain constant relation established between these, the distance of the generating point from the focus, and the perpendiculars from the latter upon the above-named right lines. The two consequent theorems may be thus expressed:

I. "*If in a plane any two circles A^2, B^2* be given, and to them, from a point X_0, tangents α, β be drawn, and it be required, that either the sum $(\alpha+\beta)$ or the difference $(\alpha-\beta$ or $\beta-\alpha)$, of these tangents shall equal a given length l, the locus of the point X_0 will always be a certain conic section C^2, making a double contact (real or imaginary) with the two circles, and of whose axes, one always coincides with the central line AB of the circles.*" And conversely, "*If in a given conic section C^2 any two circles A^2 and B^2 be described, making with it a double contact, and having their centres A and B situated in the same axis, then the tangents α, β drawn from any point X_0 of the conic section to these circles, will have a sum or difference equal to a constant length l; in fact, both these cases generally present themselves, for the arc of the conic section is divided by its points of contact with the two circles into four parts, for two of these the sum $(\alpha+\beta=l)$, for the other two the difference $(\alpha-\beta=l$ or $\beta-\alpha=l)$ is constant.*"

II. "*If in a plane any n right lines G_1, G_2, G_3......G_n together with any point A be given, and the perpendiculars x_1, x_2, x_3......x_n, let fall from a point X on these lines, be respectively divided by the perpendiculars a_1, a_2, a_3...a_n, from the fixed point A to the same lines, then the so-found quotients respectively multiplied by given coefficients α_1, α_2, α_3......α_n; and it be required that the sum of these products shall equal the line $Ax=x$, divided by a given line a,* i.e. *that*

$$\alpha_1\frac{x_1}{a_1}+\alpha_2\frac{x_2}{a_2}+\alpha_3\frac{x_3}{a_3}+\ldots+\alpha_n\frac{x_n}{a_n}=\frac{x}{a},$$

then the locus of point X will always be a certain conic section C^2 whose focus is in point A, and of which the radius of curvature r in the vertex of the principal axes can be immediately determined from the n coefficients and the length a, viz.

$$r=(\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_n)\,a;$$

similarly, the directrix G of the conic section C^2, corresponding

* Throughout, the exponent n attached to a symbol (C^n) expresses that the curve represented thereby, is of the n^{th} order.

to the focus A, depends only on the n coefficients and the fixed elements, so that when, successively, to the length a all values from 0 to ∞ are given, a group of conic sections is generated, whose constituents have in common the focus A and corresponding directrix G, and in which the ratio of the above radius of curvature to the corresponding length a is constant, i.e.

$$\frac{r}{a} = \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n."$$

If in Theorem I. the given circles A^2, B^2 are reduced to their central points A and B; and if in Theorem II. all right lines except one are disregarded, the two well-known theorems mentioned at the commencement will be obtained.

Section II.

With respect to Theorem II., I will at present merely glance briefly at one circumstance, and then subject Theorem I. to a more complete discussion.

The directrix G, inasmuch as in a certain sense it is an axis of *mean* distance in reference to the n given lines, their corresponding coefficients, and the point A, can be thus determined: If a_0 and x_0 are the perpendiculars let fall from the points A and X upon the directrix, then, for every point X in the plane,

$$\alpha_1 \frac{x_1}{a_1} + \alpha_2 \frac{x_2}{a_2} + \alpha_3 \frac{x_3}{a_3} + \ldots + \alpha_n \frac{x_n}{a_n} = (\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n) \frac{x_0}{a_0}.$$

The directrix G, however, is not thus absolutely determined. For as, in reference to each of the n given lines, the signs + and − have to be attributed to opposite sides, and as these signs can be changed at will, many different directrices and corresponding conic sections, in general 2^{n-1}, ensue from these interchanges in the same given elements (*i.e.* in the same n lines G_1, G_2, G_3 ...... G_n, the same n coefficients $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$, the same point A and the same length a).

For example, when only two lines, G_1 and G_2, are given, there are two different directrices, G and H, possible; both these pass through the intersection point of the given lines and are conjugate harmonical to them, &c. All further development of this subject I here pass over.

Section III.

I. In examining the first theorem (§ *I.* I.) more closely, we will commence with the special case, when the given circles A^2 and B^2 lie without each other, as the circles

$Ua_0U_1a_1$, and $Vb_0V_1b_1$ (see fig. 1, plate 1) described around the centres A and B, and on the segments UU_1 and VV_1 as diameters.

It is necessary to fix the following elements more closely in view, as well as to direct our attention to certain collateral circumstances.

Let a, b represent the length of the radii of the circles A^2, B^2; $2C$ the distance AB between their centres; let this distance AB be bisected in point M, i. e. $MA = MB = C$. We will call the unlimited right line $UABN$, the axis X; U and U_1, V and V_1 are the extremities of the diameters of the given circles in this axis. Further, let v and v_1 be respectively the lengths of the tangents from points V and V_1 to the circle A^2, and similarly u and u_1, the two tangents from points U and U_1 to circle B^2. Let us suppose $a > b$, then, of the four tangents, u is the greatest and u_1 the least, their order being $u > v_1 > v > u_1$. Let the line L be the so-called Line of equal Powers for the two given circles; *i.e.* the locus of the point from which two equal tangents α, β can be drawn to the two circles, so that $\alpha = \beta$ or $\alpha - \beta = 0$. Further, let R and R_1 be the outer common tangents of the two circles, and a_0 and b_0, a_1 and b_1 their points of contact; their intersection point x_0 is the outer point of similitude of the two circles.

Similarly, let S and S_1 be the inner common tangents of the two circles, α_0 and β_0, α_1 and β_1 being their points of contact; their intersection point x_1 is the inner point of similitude of the two circles. These two pairs of common tangents are so divided by their 8 points of contact, by their 4 points y_0, z_0, y_1, z_1 of mutual intersection, and by their 4 points m, μ, m_1, μ_1 of intersection with the line L, that the segments fulfil the following equations:

(1). $a_0b_0 = a_1b_1 = y_0z_1 = z_0y_1$ and $\alpha_0\beta_0 = \alpha_1\beta_1 = y_0z_0 = y_1z_1$.

(2). $a_0z_0 = b_0y_0 = b_1y_1 = a_1z_1 = \alpha_0z_0 = \beta_1y_0 = \beta_0y_1 = \alpha_1z_1$.

(3). $ma_0 = mb_0 = \mu z_0 = \mu y_1 =$ &c., and $mz_0 = my_0 = \mu\alpha_0 = \mu\beta_0 =$ &c.

Therefore the diagonals y_0y_1 and z_0z_1, of the complete quadrilateral RR_1SS_1 formed by the four common tangents, are equidistant from the line L, and together with it are perpendicular to the axis X (the third diagonal x_0x_1 of the quadrilateral) y, z, and m_0 are the intersection points of these three lines with the axis X, and $m_0y = m_0z$. The four points of contact a_0, b_0, a_1, b_1 of the outer tangents R, R_1 lie in a circle M^2, whose centre is the above-named point M.

Similarly, the four points of contact α_0, β_0, α_1, β_1 of the inner tangents S, S_1 lie in another circle M^2, concentric with last; and in like manner, the four points y_0, y_1, z_0, z_1 of alternate intersection, *i.e.* of an outer with an inner tangent, together with the centres A, B of the given circles, lie all in a third circle M_0^2, having the same centre M, and, as is evident, a radius $MA = C$. The perpendiculars let fall from the point M upon the tangents R, R_1, S, S_1, meet the same in m, m_1, μ, μ_1, respectively; these points therefore are in the line L.

Lastly, let the distance x_0x_1 between the points of similitude x_0 and x_1, be bisected in N. The circle N^2, described around N as centre with radius $Nx_0 = Nx_1 = n$, is called the circle of similitude of the given circles A^2 and B^2.

II. If, successively, to the defining length l all values from 0 to ∞ be given, the complete throng of locus-curves C^2, or $Th(C^2)$,* involved in the above theorem (§ *I.* I.) will be generated. However these curves may cover the plane, it is evident that only two of the same can pass through any point X_0 in the plane, for if α, β be the tangents from X_0 to A^2, B^2; then, for the one curve, we have $l = \alpha + \beta$, and for the other $l = \alpha - \beta$ or $= \beta - \alpha$. As will soon be shewn, it is easy to distinguish for any given length l, whether the corresponding locus-curve C^2 be an ellipse E^2, a hyperbola H^2, or a parabola P^2, as also to determine its more precise relation to the circles A^2 and B^2. The curve C^2 is an H^2, or an E^2, according as $l < AB$, or $l > AB$; from the value $l = AB = 2C$ the only parabola P^2 is obtained. With respect to their relations towards the circles A^2, B^2, the hyperbolas may be divided into three groups, and represented by $Gr(H_1^2)$, $Gr(H_2^2)$, $Gr(H_3^2)$; of these, as well as of the group of ellipses $Gr(E^2)$, the following properties must be noticed.

(1). The first group of hyperbolas $Gr(H_1^2)$ corresponds to values of l between $l = 0$ and $l = \alpha_0\beta_0$ (I.); it commences (when $l = 0$) with the line L,—which is to be considered as double, and as a hyperbola, both branches of which coincide with the second axis,—and concludes with the pair of inner

* In translating I have here, as well as in § *IX.*, had to modify the symbols used by Professor Steiner. For a throng of curves $Th(C^2)$, Prof. S. uses the symbol $S(C^2)$, (*Schaar Curven*); in § *IX.* for a pencil of curves $Pn(C^2)$, he uses the symbol $B(C^2)$ (*Curven-Büschel*). The term *Büschel* as applied to the latter implies merely that the curves constituting the *schaar* have, in this case, more properties in common with each other than before.

tangents (SS_1), when $l = \alpha_0\beta_0 = \alpha_1\beta_1$. The principal axis of every H_1^2 coincides with the axis X, and of its two branches, the one encloses the circle A^2, the other the circle B^2; at commencement, however, the contact with both circles is imaginary, until $l = u_1$ (I.), when the greater circle A^2 is touched in point U_1; this is a contact in four successive points, so that A^2 is the circle of curvature in the vertex U_1; from here onwards the H_1^2 touch the circle A^2 in two real points, but the contact with circle B^2 remains still imaginary until $l = v$, when B^2 becomes its circle of curvature in vertex V, from here on the H_1^2 give real contacts with both circles, until they reach the limit (SS_1). Consequently the points of real contact with all H_1^2 are situated along the circular arcs $\alpha_0 U_1 \alpha_1$ and $\beta_0 V \beta_1$.

(2). The second group of hyperbolas $Gr(H_2^2)$ corresponds to values of l between $l = \alpha_0\beta_0$ and $l = a_0 b_0$; it begins with the pair of inner tangents (SS_1), and ends with the outer pair (RR_1). The second axis of every H_2^2 coincides with the axis X, and each of the two branches touches both circles outwardly; all four points of contact are real throughout, and are situated in the two pairs of circular arcs $a_0\alpha_0$ and $a_1\alpha_1$, $b_0\beta_1$ and $b_1\beta_0$.

(3). The third group of hyperbolas $Gr(H_3^2)$ corresponding to values of l between $l = a_0 b_0$ and $l = AB$, commences with the outer tangents (RR_1), and concludes with the parabola P^2, which, as already stated, corresponds to the value $l = AB$, and touches the circles in certain points α and α', β and β'. Of each H_3^2 the one branch encloses both circles, and its contacts with both are real; all the principal axes are in X, and the points of contact are situated in the arcs $a_0\alpha$ and $a_1\alpha'$, $b_0\beta$ and $b_1\beta'$.

(4). Lastly, the group of ellipses $Gr(E^2)$ corresponds to values between $l = AB$ and $l = \infty$; it commences with the parabola P^2, and concludes with an ellipse E^2_∞, situated entirely in infinity. Each E^2 encloses both circles, and the principal axes of all are in X; at first they touch each circle in two real points, until the one corresponding to $l = v_1$, which makes a contact in four successive points with circle B^2, and has it as circle of curvature; from here on, the E^2 gives real contacts with the circle A^2 alone, until, when $l = u$, the corresponding E^2 makes a contact in four points with circle A^2 and has it as circle of curvature; from here to the end, all contacts are imaginary. The points of real contact of all E^2 are contained in the arcs $\alpha U \alpha'$ and $\beta V_1 \beta'$.

As, by the continuous increase of the length l, the several groups of locus-curves are successively traversed, the centre C of the curve C^2 moves along the axis X in unchanged direction, and in so doing, the centres of the several groups pass over the following segments of the axis X. In the $Gr(H_1^2)$, the centre C moves from m_0 to x_1; in the $Gr(H_2^2)$, from x_1 to x_0; in the $Gr(H_3^2)$, C moves in the same direction as before on to infinity, up to the centre C_∞ of the parabola P^2; and lastly, in the $Gr(E^2)$, C comes from infinity, from C_∞, to U, A... until at length it returns to M, exactly the centre of the last ellipse E_∞, situated entirely in infinity, and corresponding to the value $l = \infty$. Accordingly, it would appear that the centre C traverses the whole axis X, with the exception of segment Mm_0; the centres of imaginary locus-curves are situated in this segment.

Any length l being given, the points of contact of the corresponding curve C^2 with the given circles A^2 and B^2 are easily constructed. For example, to find the point of contact with the circle A^2, set off the length $b_0b_0' = l$ on any tangent R of circle B^2, from the point of contact b_0; then the auxiliary circle B_0^2, described around B as centre, with radius Bb_0', will intersect circle A^2 in the required points of contact, and should it give no real points of intersection with A^2, the contact is imaginary; in this case however the line of equal powers of circles A^2 and B^2, *i.e.* their ideal common secant, is at the same time the ideal chord of contact of C^2 and A^2. In the same manner the points of contact of C^2 with B^2 may be found. By this method, for instance, the points of contact α and α', β and β' of the parabola P^2 may be easily constructed, making $b_0b_0' = AB$. The points of contact of each of the two curves C^2 which pass through any given point X_0, may with facility be obtained in an exactly similar manner, &c.

III. The foci of the $Th(C^2)$ have a noteworthy situation, and are, in their totality, subject to an interesting law.

The circle of similitude N^2 is the locus of the foci of the second group of hyperbolas $Gr(H_2^2)$, so that the extremities of every chord of this circle, perpendicular to the diameter x_0x_1, are at the same time the foci of an H_2^2.

On the other hand, the foci of every other locus-curve are in the axis X, but so situated that the two foci of each C^2 are conjugate harmonical to the points of similitude x_0 and x_1. Accordingly, the centre N of the circle of similitude must

be the focus of the parabola P^2, because its conjugate harmonical point in reference to x_0 and x_1 is in infinity. The centres A, B of the given circles are the foci of the abovementioned special ellipse E^2_∞, for these points are conjugate harmonical to x_0 and x_1 and are equidistant from the centre M of E^2_∞.* Further, this theorem determines also the foci of the first special hyperbola H_1^2, consisting of the double line L(II. 1), for as m_0 is evidently its centre, y and z must be regarded as its foci, for they are two conjugate harmonical points to x_0 and x_1, and are equidistant from m_0, $ym_0 = zm_0$ (I.).

Accordingly the foci of all locus curves fulfil the following common condition.

"*The rectangle under the distances of the two foci* (f *and* f_1) *of each locus-curve* C^2 *from the point* N (*the focus of the parabola* P^2 *or centre of circle of similitude* N^2) *is constant and* $= n^2$, i.e. *equal to the square of the radius of the circle of similitude; hence throughout we have* $fN.f_1N = n^2$."

From this, if the centre C of any locus-curve C^2 be given, its foci f and f_1 are easily determined and constructed. For if C be in the diameter x_0x_1, then, as already stated, the extremities of the chord of circle N^2, erected in C perpendicular to diameter x_0x_1, are the required foci. On the other hand, if C be situated in the production of the diameter on either side, then the tangent, drawn from C to the circle N^2, is equal to the eccentricity of the curve C^2, so that the circle described around C as centre, with this tangent as radius, intersects the axis X in the required foci f and f_1. Hence also from this the axes of the curve C^2, as well as the corresponding length l, may be determined. For let the eccentricity already found $Cf = Cf_1 = \gamma$, and let α, β be the semi-axes of the curve, a and b being, as before, the radii of circles A^2 and B^2; then, in the $Gr(H_2^2)$,

$$\alpha : \gamma = a : Af = b : Bf.$$

In the other groups, however,

$$\beta^2 : \gamma^2 = a^2 : Af.Af_1 = b^2 : Bf.Bf_1.$$

By the former, α is immediately found; by the latter, at first only β^2; in both, the remaining axis may be found from the known relation between α, β and γ. The length l

* In every common ellipse the circle of double contact, if its centre is in a focus, is reduced to a point, *i.e.* its radius $= 0$ (see *Crelle's Journal*, vol. XXXVII. p. 175). The above special ellipse E^2_∞, whose periphery is in infinity, is therefore, in this respect, an exception.

is defined by the proportion

$$l : AB = \alpha : \gamma.$$

IV. Let the points in which any locus-curve C^2 touches the circles A^2 and B^2 be respectively p and p_1, q and q_1.

"*The chords of contact pp_1 and qq_1 are parallel to the line L, are always equidistant from, and like it, perpendicular to the axis X, (this too holds true when the contact is imaginary and the chords are ideal).*"* And conversely, "*Every two lines equidistant from, and parallel to L, are the chords of contact of some locus-curve C^2 with the circles A^2 and B^2.*" Further, "*The tangents β drawn from points p and p_1 to circle B^2 are each equal to the tangents α drawn from q and q_1 to circle A^2, and both lines are equal to the length l corresponding to the curve C^2.*"

"*The four points of contact p, p_1, q, q_1 are always in the circumference of a circle M^2, whose centre is the point M.*" And conversely, "*Every circle M^2, described around M as centre, intersects the given circles A^2 and B^2 in four points, which will be the points of contact of some locus-curve C^2 with them.*"

"*The eight points, in which the given circles are touched by any two locus-curves, lie in a certain third conic section D^2.*" For example, the eight points of contact a_0, a_1, b_0, b_1, and $\alpha_0, \alpha_1, \beta_0, \beta_1$, of the two pairs of common tangents R, R_1 and S, S_1, lie in a certain conic section D^2. And conversely, "*If through the four points of contact p, p_1, q, q_1 of any curve C^2, any conic section D^2 be drawn, it will intersect the circles A^2 and B^2 in four new points p° and p_1°, q° and q_1°, which will likewise be the points of contact of some other locus-curve C_1^2 with the circles.*"

"*The four points in which any two locus-curves C^2 and C_1^2 intersect one another, lie always in a certain circle M^2 around M as centre.*" And conversely, "*Every circle M^2 around M as centre, cuts any locus-curve C^2 in four points, which are at the same time the points wherein some other curve C_1^2 also intersects the same curve C^2.*"

Hence, as a special case, every curve C^2 intersects the tangents R and R_1 in four points, which lie in a certain circle M^2; and further, these tangents determine equal chords in the curve C^2: the same applies also to the inner tangents S and S_1, indeed further:

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* This theorem is also to be found in the memoir above referred to (*Crelle's Journal*, vol. XXXVII. p. 176).

"*The four tangents R, R_1, S, S_1 determine in every locus-curve C^2 four equal chords, and in fact, these chords are each equal to the corresponding length l, and are all bisected by the line L in the points m, m_1, μ, μ_1.*" Consequently, any length l being given, the eight points in which the corresponding locus-curve C^2 intersects the four tangents R, R_1, S, S_1, may be easily found.

If the two pairs of points of contact, p and p_1, q and q_1, of any locus-curve C^2 be joined cross-wise by the right lines pq, pq_1, p_1q, p_1q_1, which may be called *Alternate Chords*, then all alternate chords have the following common property.

Every alternate chord determines, in the given circles, equal chords; i.e. *if, for example, the line pq intersects the circles A^2 and B^2 a second time in the points p° and q°, then chord $pp^\circ = qq^\circ$ for all positions of p and q.* Further,

"*The line L is the locus of the middle m° of every alternate chord, and the perpendicular let fall from M upon any chord meets it in this same bisection point m°. Hence, all alternate chords are tangential to a certain parabola P_0^2, of which M is the focus, and L the tangent in vertex m_0 of its axis, and which, in particular, has the four tangents R, R_1, S, S_1 in common with the circles A^2 and B^2* (for these are but special alternate chords)."

Let P and P_1, Q and Q_1 be the common tangents of circles A^2 and B^2 and curve C^2; *i.e.* the tangents which, in the points p and p_1, q and q_1, touch both the curve C^2 and the circles A^2 and B^2 respectively. These four tangents have analogous properties to the four points; I will here, however, mention but a few, leaving the rest for after consideration, when instead of the circles A^2 and B^2 any conic sections are given.

Let r be the intersection point of P and P_1, and r_1 that of Q and Q_1; further, let s and s_1, t and t_1, be respectively the points of alternate intersection; *i.e.* of P with Q and P_1 with Q_1, of P with Q_1 and P_1 with Q, so that, consequently, r and r_1, s and s_1, t and t_1 are opposite corners of a complete quadrilateral PP_1QQ_1.

"*The points r and r_1 are in the axis X, and always conjugate harmonical to the points of similitude x_0 and x_1.*"

"*The circle of similitude N^2 is the locus of all alternate intersection points s, s_1, t, t_1.*"

It must be here noticed, that the tangents P and P_1 will, in one of their positions, coincide with the outer common tangents of circles N^2 and A^2, and will then touch N^2 in

certain points t° and t_1°; in another of their positions they will coincide with the inner common tangents of the same two circles, and will then touch N^2 in certain points s° and s_1°; *in each of these cases the chord of contact $t^{\circ}t_1^{\circ}$ or $s^{\circ}s_1^{\circ}$ touches the circle B^2 in points V_1 or V*, for both tangents Q and Q_1 will simultaneously coincide with the corresponding chord, and the curve C^2 will make a contact in four consecutive points with circle B^2 in the respective point V_1 or V. (II.). Conversely, "*If to any two circles A^2 and N^2, situated without each other, the two pairs of common tangents be constructed, and in either circle,* e.g. *N^2, the two chords of contact, $t^{\circ}t_1^{\circ}$ and $s^{\circ}s_1^{\circ}$, of the pair of tangents be drawn; afterwards, on the segment V_1V which these chords of contact determine on the axis X, a third circle B^2 be described, then the circle N^2 will be the circle of similitude to the circles A^2 and B^2.*"

Section IV.

If the given circles A^2 and B^2 intersect each other, or the one be situated entirely within the other, the properties established in § *III.* suffer some modification, or new circumstances present themselves, to embrace which, the conditions (§ *I.* I.) for the generating point X_0 must be more generally defined. This more general definition results from a consideration of the powers of the point X_0 in reference to the circles.* As the power of a point X_0 in reference to a circle A^2 can be an outer or an inner one, and as such, represented either by the square of the tangent α from it to the circle, or by the square of half the smallest chord α_1 through it, according as the point is situated *without* or *within* the circle; so also, when two circles A^2 and B^2 are given, the locus of point X_0 may be required, for which the sum $\alpha_1 + \beta_1$, or the difference, $\alpha_1 - \beta_1$ or $\beta_1 - \alpha_1$, of half the smallest chords which can be drawn through it in the given circles, shall equal a given length l. This condition can, however, be combined with the above, with respect to the tangents α and β, in the more embracing problem—

"*To find the locus of the point X_0, for which the* (*square*) *roots of the like powers in reference to the given circles, A^2 and B^2, shall have either a sum or a difference equal to a given line l.*"

If, further, the loci for inner and outer powers, considered separately, correspond to the same value l, it is true that by each method a different conic section will be produced,

* See *Crelle's Journal*, vol. I. p. 163.

yet the two will have a certain connexion, and may be considered as naturally complementary to each other. In a similar manner the locus of point Y_0 may also be required, for which the sum or difference of the *unlike-named* powers (*i.e.* the tangent to one circle, and half the smallest chord in the other,) shall be equal to a given length l. In this case, however, the required locus is in general a curve of the fourth degree.

With reference to the above, the foregoing properties suffer the following modifications, when the mutual position of the given circles undergoes the above-described changes.

Section V.

I. If we allow the two circles A^2 and B^2 (fig. 1, plate I.) to approach each other until, meeting in points U_1 and V, they there touch each other in one point (U_1V); then the two inner tangents S and S_1 will coincide with the line L, which will then be the common tangent of the circles in the point (U_1V), with which point, too, the inner point of similitude x_1, together with many others, will now coincide. Consequently, the first group of hyperbolas $Gr(H_1^2)$(§ *III.* II.) vanishes, inasmuch as its last constituent (SS_1) combines with its first L, or in other words, is reduced to the single constituent l, with which at the same time the second group $Gr(H_2^2)$ commences. This group, as before, ends with the pair of outer tangents (RR_1), and it, together with all other groups, remains the same as before.

II. If the circles A^2 and B^2 cut each other, as in fig. 2, the line L will pass through their points of intersection r and s, and through the same points the circle of similitude $N^2 = x_0rx_1s$ will pass. The $Gr(H_2^2)$ begins here, too, with the line L, and ends with (RR_1), their foci, however, no longer occupy the whole circumference of the circle of similitude N^2, but only its arc rx_0s. The groups $Gr(H_3^2)$ and $Gr(E^2)$ retain their former properties (§ *III.* II.). On the other hand, a new group of ellipses $Gr(E_1^2)$ now appears, determined by the inner powers, (by the semi-chords α_1 and β_1) and situated within both circles, in the curvilinear two-cornered figure $rVsU_1r$; these ellipses are enclosed and touched twice by each circle, the second or minor axis of each E_1^2 coincides with the axis X. This $Gr(E_1^2)$ is, in a certain sense, to be considered as the continuation of $Gr(H_2^2)$, that is to say, the transition takes place through the line L, which belongs to both groups, inasmuch as the segment rs must be considered as an E_1^2; and on the other

hand, its two infinitely long segments beyond r and s, as the two branches of an H_2^2, for both correspond to the same value of l, viz. $l = 0$, or respectively $\alpha_1 = \beta_1$ and $\alpha = \beta$; in both, therefore, the two points r and s are to be considered as, at the same time, the foci and principal vertices. Consequently, the loci of the foci of both groups are in intimate connexion with each other, just as the foci of the $Gr(H_2^2)$ are all contained in the arc rx_0s, so also are the foci of the $Gr(E_1^2)$ in the arc rx_1s, of the circle of similitude N^2; hence the extremities of every chord of the arc rx_1s, perpendicular to the segment m_0x_1, are at the same time the foci of an E_1^2. According to this, the centres of the $Gr(E_1^2)$ are all contained in the segment m_0x_1. If the length l be allowed to increase from $l = 0$, the centre E_1 of the corresponding locus-curve E_1^2 moves from m_0 to x_1; here $l (= \alpha_1 + \beta_1)$ attains a certain maximum limit, and the curve is reduced to its centre x_1. In this case, that is, when the locus of point X_0 is restricted to a single point x_1, the said maximum is represented by half the smallest chords passing through x_1, both of which are contained in the line $\alpha_0x_1\beta_0$, perpendicular to the axis x, so that $\alpha_0x_1 + \beta_0x_1 = \alpha_0\beta_0$ is the maximum limit of l. Hence, "*Among all the points within both circles A^2 and B^2, the inner point of similitude x_1 has the property of making the sum of the smallest chords through it a maximum.*"

When the length l is given, the points p and p_1, q and q_1, in which any inner locus-curve E_1^2 touches the circles A^2 and B^2, can be constructed in an analogous manner to the one before mentioned (§ *III.* II.). If, for instance, in the circle B^2, a chord of the length $2l$ be drawn, and bisected in m, then the circle described around B as centre, with radius Bm, will cut the circle A^2 in the required points of contact p and p_1. Further, the limits, where real contacts cease, can be determined in an analogous manner. If v be half the smallest chord in circle A^2 through point V, and u_1 half the smallest chord in circle B^2 through point U_1, then the contacts of curve E_1^2 with circles A^2 and B^2 are only real as long as l remains, respectively, smaller than u_1 and v; if $l = u_1$ or $l = v$, the contact with one of the circles in U_1 or V will be in four consecutive points; hence it will be the circle of curvature to the respective locus-curve. If radius $a > b$, then $v > u_1$, and the imaginary contacts with circle A^2 begin earlier than those with B^2. Conceive a curve E_1^2, which touches the circles A^2 and B^2 in real points p and p_1, q and q_1, it will then be evident, that for all points X_0 in the

elliptic arcs situated between the points of contact of different circles, *i.e.* in the arcs pq and p_1q_1, the sum $\alpha_1 + \beta_1 = l$. On the contrary, for points in the elliptic arcs pp_1 and qq_1, between the points of contact of the same circle, the differences $\beta_1 - \alpha_1$ and $\alpha_1 - \beta_1$ will be respectively equal to the constant length l. When the points p and p_1 are imaginary, *i.e.* if $l>u$, but $<v$, then E_1^2 will be divided by points q and q_1 into two arcs, of which the one situated nearest to point U_1 corresponds to the sum $\alpha_1 + \beta_1$; on the contrary, the one nearest point V corresponds to the difference $\alpha_1 - \beta_1$. If all four contacts are imaginary, then for all points X_0 in E_1^2, the sum $\alpha_1 + \beta_1 = l$. Similar remarks might have been made above (§ *III.* II.), with reference to the $Gr(E^2)$, and the unlike properties of the arcs of the several groups of hyperbolas in this respect may, with facility, be more particularly defined.

III. If the centres of the circles A^2 and B^2 approach each other still more, so that points V_1 and U_1 coincide, and the circles touch each other, but only in one point (V_1U_1), the two outer common tangents will likewise coincide with line L, which will be a tangent to both circles in point (U_1V_1): it is also to be considered as the last remnant of the second group of hyperbolas $Gr(H_2^2)$, now also vanished, and at the same time as the commencing constituent of the third group $Gr(H_3^2)$. The outer point of similitude x_0, together with points of intersection r and s, now coincide with point (U_1V_1), so that the circle of similitude N^2 touches the given circles in this point. The inner group of ellipses $Gr(E_1^2)$ will here be more complete, their foci will occupy the whole circumference of the circle of similitude, and their centres its diameter x_0x_1. The first constituent of $Gr(E_1^2)$, corresponding to the value $l = 0$, is identical with the point (U_1V_1); no other E_1^2, except it, forms a real contact with circle A^2, just as no other locus-curve of remaining groups, the line L excepted, can make a real contact with circle B^2. Similarly, the last constituent of $Gr(E_1^2)$ is reduced to the point x_1, which corresponds, as above (II.), to the maximum limit of l.

IV. Lastly, when circle B^2 is situated entirely within the circle A^2 (fig. 3), the line L will be situated at a certain distance without both circles, whereas the points of similitude x_0 and x_1, and hence also the circle of similitude, are within the circle B^2. Here the $Gr(H_3^2)$ still exists; it commences with the line L and the value $l = 0$, and concludes with the value $l = AB$, corresponding to the para-

bola P^2; at the same time, the latter is the commencement of $Gr(E^2)$ which ends, as before, with E^2_∞(§ 3, II.). On the other hand, with respect to the interior locus-curves, the $Gr(E_1^2)$ begins with the outer point of similitude x_0, and in fact with a value of l corresponding to the minimum of the difference $\alpha_1 - \beta_1$. This follows from the following theorem: "*Among all points X_0 within the circle B^2, the outer point of similitude x_0 has the property of making the difference of the smallest chords, $2\alpha_1$, $2\beta_1$, and through it, a minimum.*" The line $\alpha_0\beta_0 x_0$ perpendicular to the axis X in point x_0, contains these two particular chords, so that $x_0\alpha^\circ - x_0\beta^\circ = \alpha^\circ\beta^\circ$ is exactly the value of l, for which the first E_1^2 is reduced to the point of similitude x_0. Similarly, the last constituent of $Gr(E_1^2)$ is reduced to the inner point of similitude x_1, and corresponds to that value of l which is the maximum of the sum $\alpha_1 + \beta_1$, and, as before, (II.) is represented by $x_1\alpha_1^\circ + x_1\beta_1^\circ = \alpha_1^\circ\beta_1^\circ$, in the line $\alpha_1^\circ x_1\beta_1^\circ$ perpendicular to axis X in the point x_1. In the $Gr(E_1^2)$, therefore, the length l varies between the limits $l = \alpha^\circ\beta^\circ$ and $l = \alpha_1^\circ\beta_1^\circ$.

In the present situation, the circle A^2 forms real contacts with the exterior locus-curves $Gr(H_3^2)$ and $Gr(E^2)$ alone; on the contrary, the circle B^2 only with the interior $Gr(E_1^2)$. The limits where, in both cases, the real contacts begin and end, may be determined in the above-mentioned manner; and similarly, a length l being given, the points of contact can be easily constructed by the method already explained. A special circumstance with respect to the exterior locus-curves, shall here be more particularly considered.

Whether any of the $Gr(H_3^2)$ attain to real contact with the circle A^2 or not, depends upon whether $u_1 < AB$ or $u_1 > AB$; *i.e.* whether the tangent u_1(§ *III.* I.), from the point U_1 (which, of all points in A^2, is nearest the circle B^2) to the circle B^2, is less or greater than AB. If $u_1 = AB$, the last constituent alone of the $Gr(H_3^2)$, *i.e.* the parabola P^2, makes in U a real contact in four points with the circle A^2. If, on the other hand, $u > AB$, then, after P^2, follow a certain number of ellipses in the $Gr(E^2)$, which form no real contact; these, for distinction, we will represent by $Gr(E^2_-)$. For all points X_0 in such an ellipse E^2_-, the difference only, $\beta - \alpha = l$, (the same remark applies also, in this case, to every H_3^2). The $Gr(E^2_-)$ corresponds to values between $l = AB$ and $l = u_1$. When $l = u_1$ the corresponding ellipse makes a contact in four points with circle A^2 in U_1, and, for its whole circumference, the difference $\beta - \alpha$ yet equals l; but at the

same time it is the commencement of the group of ellipses with real points of contact. From here on, as l increases, the E^2 touch the circle A^2 each in two real points p and p_1, by which their arcs are divided each in two parts; of these, the one which overspans U_1 corresponds to the sum $\alpha + \beta$, and the other over U, to the difference $\beta - \alpha$. The value $l = u$ (tangent from U to B^2) corresponds to the last E^2 whose contact with A^2 is real; in u that contact is one in four consecutive points, and for all points in it, the sum alone $\alpha + \beta = l$. From here on, by the increase of l up to $l = \infty$, a new section of ellipses are generated in the $Gr(E^2)$, which we may represent by $Gr(E^2_+)$; they also form no real contact with circle A^2, but for every point in their peripheries the sum $\alpha + \beta$ alone is made equal to the constant length l. Consequently, under the supposition that $u_1 > AB$, the $Gr(E^2)$ contains two distinct sub-groups, $Gr(E^2_-)$ and $Gr(E^2_+)$, both of which enclose, and give imaginary contacts with circle A^2, but are nevertheless essentially different from each other, inasmuch as the points of first make only the difference $\beta - \alpha$ constant, whilst those of the second make only the sum $\alpha + \beta$ constant. These different properties will be explained by the following nearer relation of the two circles to the respective curves. Let, generally, f and f_1 be the foci of an ellipse, and k and k_1 the centres of curvature of the vertices in its major axis; the two last points are situated between the two first, let k be nearest f, and k_1 nearest f_1. Then the segments fk and f_1k_1 will contain the centres of all circles which make two imaginary contacts with the ellipse.* Hereby, the relations of $Gr(E^2_-)$ and $Gr(E^2_+)$ towards the given circles can be thus more strictly defined:

"*In every ellipse E^2_- the centres, A and B, of the circles lie both in the same segment fk or f_1k_1; whereas in every ellipse E^2_+ these centres are situated in different segments, the one in fk and the other in f_1k_1.*"

From the preceding we may, lastly, deduce the following theorem:

"*The locus-curves to two circles A^2 and B^2, when the one is situated within the other, can only contain the description E^2_- (for all points of which the difference only $\beta - \alpha = l$) when $u_1 > AB$; and thereby, the length l varies between the limits $l = AB$ and $l = u_1$.*" And conversely, "*If in a given ellipse two circles are described, so that their double contact with the*

* See *Crelle's Journal*, vol. XXXVII. p. 175.

ellipse is imaginary, both their centres being situated in the same segment fk or f_1k_1; then, for all points X_0 of the ellipse, the difference $\beta - \alpha = l$ is constant, and u_1 is always greater than AB; the constant l, however, greater than AB, but smaller than u_1."

Section VI.

From the foregoing considerations it is easy to infer that if, in a plane, any three circles A^2, B^2, and D^2 be given, whose centres A, B, and D are in the same right line X, in general a certain conic section C^2 exists, which with reference to each two circles, shall be one of their corresponding locus-curves, and which, as a consequence, will form a double contact with each circle. The length l corresponding to each pair of circles can, for example, be thus determined.

Let a, b, and d be the respective radii of the circles, and the distance of their respective centres be thus represented, $AB = 2\gamma$, $AD = 2\gamma_1$, and $BD = 2\gamma_2$; and further, let the length l, corresponding to the pairs of circles A^2 and B^2, A^2 and D^2, B^2 and D^2, be respectively 2λ, $2\lambda_1$, $2\lambda_2$; then, if B lies between A and D, we have the following relations:

$$\lambda^2 = \frac{\gamma}{\gamma_1\gamma_2}(4\gamma\gamma_1\gamma_2 - \gamma_2 a^2 + \gamma_1 b^2 - \gamma d^2),$$

$$\lambda_1^2 = \frac{\gamma_1}{\gamma\gamma_2}(4\gamma\gamma_1\gamma_2 - \gamma_2 a^2 + \gamma_1 b^2 - \gamma d^2),$$

$$\lambda_2^2 = \frac{\gamma_2}{\gamma\gamma_1}(4\gamma\gamma_1\gamma_2 - \gamma_2 a^2 + \gamma_1 b^2 - \gamma d^2).$$

Section VII.

If, further, the locus of the points Y_0 were required, for which, in reference to two given circles A^2 and B^2, the roots of the unlike-named powers shall have a sum ($\alpha+\beta_1$ or $\beta+\alpha_1$), or a difference ($\alpha-\beta_1$, $\beta_1-\alpha$, or $\beta-\alpha_1$, $\alpha_1-\beta$), equal to a given length l (§ *IV.*), (whereby therefore the point Y_0 must necessarily be always within one circle and without the other), it would be found that, in general, this locus is a curve of the fourth degree $= C^4$, which touches each of the two circles in four points (real or imaginary); these are easily constructed by means of the concentric auxiliary circles (B_0^2 and A_0^2), as above explained (§ *III.* II., and § *V.* II.).

When, however, as a special case, $l = 0$,—*i.e.* if merely the locus of a point Y_0 is required, which, in reference to the

two circles, has unlike-named, but equal powers, $\alpha = \beta_1$ or $\beta = \alpha_1$,—then the curve C^4 reduces itself to a doubled circle, inasmuch as the two parts, of which it in general consists, now coincide and form a single circle C_0^2. This circle C_0^2 is also thus defined: its centre is the point M, the middle of AB; and it has, in common with the given circles, the line L as its line of equal powers. If, therefore, the given circles A^2 and B^2 cut each other, as in fig. 2, C_0^2 passes through the points of intersection r and s; if B^2 is completely within A^2, as in fig. 3, then C_0^2 is situated in the space between B^2 and A^2; and lastly, if A^2 and B^2 be without each other, as in fig. 4, but so that M falls within A^2, then the circle C^2 can yet be real, and will be entirely within A^2. From these properties the following theorem may be deduced:

"*The locus of the point having equal but unlike-named powers with respect to two given circles A^2 and B^2, is a certain third circle C_0^2, whose centre M is the middle of the line AB, joining the centres of the given circles; and, in common with these circles, C_0^2 has the line L as its line of equal (and like-named) powers.*"

The same theorem may be somewhat differently expressed thus:

"*The locus of centre Y_0 of the circle Y_0^2, which is intersected by one of the given circles A^2 or B^2 (no matter which), perpendicularly, and by the other in a diameter, is a certain fourth circle C_0^2, whose centre M is in the middle of AB, and which has a secant L (real or ideal) in common with the given circles.*"

Further, if the given circles A^2 and B^2 are concentric, the curve C^4, for every given length l, reduces itself to two circles C^2 and C_1^2, also concentric with the given circles; their radii c and c_1 always fulfil the equation

$$c^2 + c_1^2 = a^2 + b^2,$$

i.e. ***the sum of the squares of these radii is constant, and equal to the sum of the squares of the radii of the given circles A^2 and B^2***, with which latter, indeed, the circles C^2 and C_1^2 coincide, when $l = u = u_1$ (§ *III.* 1.). When $l = 0$, the circles C^2 and C_1^2 coincide with each other and form the above-mentioned C_0^2, for whose radius c_0 we have the equation

$$2c_0^2 = a^2 + b^2.$$

Section VIII.

The above considerations revealed the existence of an infinite throng of curves of the second order, $Th(C^2)$, which possess the property of forming two contacts with the given circles A^2 and B^2; nevertheless, all the conic sections possessing this same property are not contained therein; on the contrary, there are in general two more throngs possessing a like property. With respect to these latter, the following particulars are worthy of notice.

The given circles (indeed every two conics in the same plane) have, in common with each other, a triplet of conjugate poles x, y, and z, as well as a triplet of conjugate polars X, Y, and Z; the former are the angles, the latter their respective opposite sides, of one and the same triangle. One of these poles x is in infinity, and in fact, in the direction of line L, of which it is to be considered as the infinitely distant point; this pole is always real, whereas both the others, y and z, are simultaneously *imaginary* or *real*, according as the circles *do* or *do not* intersect each other; for they are at the same time the intersection points of the axis (or polar) X with every circle which intersects both the given circles, A^2 and B^2, perpendicularly; or, provided the circles are without each other, as in fig. 1, the poles y and z are also the respective intersection points of the diagonal $x_0x_1 = X$ with the two remaining diagonals $z_0z_1 = Z$ and $y_0y_1 = Y$, of the quadrilateral RR_1SS_1, formed by the four common tangents. The three throngs of conic sections already mentioned have the following essential relations towards these three poles.

The first locus-curves $Th(C^2)$ have reference to the pole x, and shall therefore be represented by $Th(C_x^2)$, for the chords of contact pp_1 and qq_1 of every curve C_x^2 are parallel to the line L, and hence, with it, directed towards the pole x (§ *III.* IV.). A second throng of conic sections $Th(C_y^2)$ touch each of the given circles twice, and have reference, in a similar manner, to the pole y, inasmuch as the chords of contact, pp_1 and qq_1, of every curve C_y^2 pass through this pole. Similarly, a third throng of conic sections $Th(C_z^2)$ also touch each of the given circles twice; their chords of contact, however, always pass through the pole z. The following are some of the interesting properties possessed by these two last throngs of conic sections.

(1) "*The chords of contact, pp_1 and qq_1, of every curve C_y^2, as well as of every curve C_z^2, are always perpendicular to each*

other; and conversely, *If through the pole y or z, any two secants pp_1 and qq_1 be drawn perpendicular to each other, they will intersect the two circles A^2 and B^2 respectively, in certain points p and p_1, q and q_1, which will be the points of contact of some curve C_y^2 or C_z^2."*

(2) "*Of the two axes of each curve C_y^2 or C_z^2, the one passes through centre A, the other through B. Consequently, the circle $M_\circ^2$, on AB as diameter (§ III. I.), is the locus of the centres of the $Th(C_y^2)$, as well as the $Th(C_z^2)$; so that every point of this circle is at the same time the centre of a curve C_y^2, as well as of a curve C_z^2; and hence, the axes of both these curves coincide.*"

(3) "*The individual curves of the $Th(C_y^2)$, as also of the $Th(C_z^2)$, are, among themselves, similar; and, in fact, the squares of the axes of each C_y^2 have to each other the same ratio as the distances of the pole y from the centres A and B; and similarly, the squares of the axes of each C_z^2 are to each other as the segments zA and zB,* viz. *thus, if α, β be the semi-axes of a C_y^2, of which α passes through A, and β through B; then*

$$\alpha^2 : \beta^2 = yB : yA;$$

and, similarly, if α_1, β_1 be the semi-axes of a C_z^2, passing respectively through A, B; then

$$\alpha_1^2 : \beta_1^2 = Bz : Az.$$

But as y and z are conjugate poles with respect to circles A^2 and B^2, so that

$$Ay.Az = a^2, \text{ and } Bz.By = b^2,$$

it follows from the above proportions, that

$$\alpha\alpha_1 : \beta\beta_1 = b : a;$$

i.e. *with respect to every two curves C_y^2 and C_z^2, whose axes coincide in direction, the rectangle under the axes passing through A, is to the rectangle under the axes passing through B, as the radius b of the circle B^2 is to the radius a of the circle A^2.*"

(4) "*The locus of the foci of each of the two throngs,* e.g. *of $Th(C_y^2)$, consists, in general, of two circles A_y^2 and B_y^2 respectively concentric with A^2 and B^2, and intersecting each other either perpendicularly or in a diameter. If the principal axis of a curve C_y^2 passes through A or B, its foci f and f_1 are respectively in the circle B_y^2 or A_y^2. The rectangle under the distances of each pair of foci, f and f_1, from the point A, as*

well as from the point B, is constant, and equal to the square of the radius a_y or b_y of the corresponding circle A_y^2 or B_y^2; hence

$$Af.Af_1 = a_y^2, \text{ and } Bf.Bf_1 = b_y^2.$$

Similarly, the foci of the $Th(C_z^2)$ are situated in two circles A_z^2 and B_z^2, which possess analogous properties."

(5) "*If between the two pairs of points p and p_1, q and q_1, in which each curve C_y^2 touches the given circles A^2 and B^2, the four alternate chords pq, pq_1, p_1q, and p_1q_1 be drawn, all such chords will be tangential to a certain conic section Y^2, which has the pole y as focus, and, in common with circles A^2 and B^2, the four (real or imaginary) tangents R and R_1, S and S_1; its foci y and (the yet unknown one) y_1 are conjugate harmonical points to A and B. Each alternate chord determines in the circles A^2 and B^2 two chords s and s_1; the ratio of these chords is for all alternate chords the same,* i.e. $s : s_1 = k$ *constant. Similarly, the alternate chords of the $Th(C_z^2)$ are all tangential to a certain conic section Z^2, of which pole z is a focus, and which also has, in common with circles A^2 and B^2, the same four tangents; its foci z and z_1 are conjugate harmonical points to A and B. Here again the alternate chords determine, in the given circles, certain chords s and s_1, whose ratio is constant, though different from the preceding,* e.g. $s : s_1 = k_1$ *constant.*"

(6) "*If P and P_1, Q and Q_1 be the tangents common to circles A^2 and B^2, and curve C_y^2* (§ *III.*, IV.) *the intersection points $PP_1 = r$ and $QQ_1 = r_1$ are always situated in the polar Y, and, in their totality, the pairs r and r_1 form a System of Points (Involution). On the other hand, the locus of the four alternate intersections PQ and P_1Q_1, PQ_1 and P_1Q, or s and s_1, t and t_1* (§ *III.* IV.), *is a certain circle N_y^2, which passes through the same pair of opposite corners, y_0 and y_1, as Y; the line L is common secant to it, and to the circles A^2 and B^2, so that its centre is also in the axis X. In this respect the $Th(C_z^2)$ also has exactly analogous properties.*"

To shew the influence of the several relative situations of the given circles on the above properties, we will examine more closely their most essential positions; that is, when they are entirely external to each other, and when B^2 is completely within A^2. The $Th(C_y^2)$ and $Th(C_z^2)$ for the intermediate position, where the circles intersect each other, are imaginary.

I. "*If the circles are without each other, as in* fig. 1, *both* $Th(C_y^2)$ *and* $Th(C_z^2)$ *consist of hyperbolas* $Th(H_y^2)$ *and* $Th(H_z^2)$*; the constituents of each throng are, among themselves, similar. The circles* A^2 *and* B_y^2*, described around the points* A *and* B *as centres, and which contain the foci of* $Th(C_y^2)$*, intersect each other perpendicularly in the opposite corners,* y_0 *and* y_1*, of the quadrilaterals* R, R_1, S, S_1*; and similarly, on the other hand, the circles* A_z^2 *and* B_z^2 *intersect each other perpendicularly in the corners* z_0 *and* z_1*. If the centre of an* H_y^2 *is in the arc* $y_0z_0Az_1y_1$ *of the circle* M_0^2*, the curve itself surrounds the circle* B^2*, and hence its principal axis passes through* B *and intersects the circle* A_y^2 *in the foci* f *and* f_1*. On the contrary, if the centre of an* H_y^2 *is in the arc* y_0By_1*, it surrounds the circle* A^2*, its principal axis passes through* A*, and its foci are in the circle* B_y^2*. The transition from the one section to the other takes place through the pair of tangents* (RS_1) *and* (R_1S)*, which are special* H^2*, and have respectively* y_0 *and* y_1 *as centres. The hyperbolas* H_z^2 *have exactly similar properties. The asymptotes of every* H_y^2 *pass through the fixed corner points* z_0 *and* z_1*; and similarly, the asymptotes of every* H_z^2 *pass through* y_0 *and* y_1*.*"

II. "*If circle* B^2 *is entirely within* A^2*, as in* fig. 3, *both throngs* $Th(C_y^2)$ *and* $Th(C_z^2)$ *consist of ellipses* $Th(E_y^2)$ *and* $Th(E_z^2)$*. Every* E_y^2 *encloses the circle* B^2*, and is enclosed by the circle* A^2*; hence its principal axis passes always through the point* B*, and its foci* f *and* f_1 *are always in a certain circle* A_y^2*, around* A *as centre; (here the circle* B_y^2 *around* B *will be intersected by circle* A_y^2 *in a diameter, but these same intersection points are the only real foci it contains). Similarly, every* E_z^2 *encloses circle* B^2 *and is enclosed by* A^2*, so that its principal axis passes only through* B*, and its foci are all contained in a certain circle* A_z^2 *around* A *as centre.*"

Section IX.

NOTE.—In the foregoing, three examples incidentally presented themselves, where the locus of a right line (there called alternate chord, § *III.* IV. and § *VIII.* V.), determining in the given circles A^2 and B^2 certain chords s and s_1, of constant ratio to each other, was found to be a conic section This property is a general one, and furnishes the following theorem:

" *The locus of a line* G *which intersects two given circles* A^2 *and* B^2*, so that the chords* s *and* s_1 *thereby formed have to each other a constant ratio* k, i. e. $s:s_1=k$, *is always a certain conic*

section G^2. All conic sections thus generated, provided the value k assumes successively all magnitudes, form a pencil of curves $Pn(G^2)$ with four (real or imaginary) common tangents (R, R_1, S, S_1), and, in fact, the given circles A^2 and B^2 themselves belong to this pencil, for they correspond respectively to the values $k=0$ and $k=\infty$. As above* (§ *III.* IV.), *the parabola $P_0^2(=G^2)$, of which the point M is focus, and the line L tangent in vertex, corresponds to the value $k=1$ or $s=s_1$. The two points of similitude x_0 and x_1 together form a special G^2 corresponding to the value $k=a:b$, &c...."* And conversely, "*The tangents of every conic section G^2, which has four real or imaginary tangents in common with two circles A^2 and B^2, determine, in these circles, certain chords s and s_1, whose ratio to each other is constant,* i. e. *for all tangents this ratio has a certain value k, &c...."*

Instead of a full discussion of this theorem, the following few remarks must here suffice.

The centres of the locus-curves $Pn(G^2)$ are all in the axis X, with which, too, one axis of the curve always coincides. Whether this same axis coinciding with X be the first or the second, depends upon whether its centre is situated without or within the segment AB. Hence the curves may be divided into two groups $Gr(G_1^2)$ and $Gr(G_2^2)$. The foci of these two groups fulfil the following conditions :

"*The foci of the $Gr(G_1^2)$ are in the axis X, and each pair are conjugate harmonical points to A and B. On the other hand, the foci of the $Gr(G_2^2)$ are in the circle M_0^2, whose diameter is the segment $AB=2c$* (§ *III.* I.), *so that each pair of foci forms, at the same time, the extremities of a chord of the circle M_0^2, perpendicular to this diameter AB.*"

From this it follows, as above, (§ *III.* III. and § *VIII.* IV.), that both groups fulfil the common condition; viz.

"*The rectangle under the distances, f and f_1, of each curve G^2 from the point M, the focus of the parabola P_0^2, is constant and equal to c^2.*"

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* As early as 1827 I forwarded this, together with several other theorems, to the Editor of the *Annales des Mathématiques*, at Montpellier; afterwards, probably by mistake, he allowed them to be published in another name.

NOTE ON THE MECHANICAL ACTION OF HEAT, AND THE SPECIFIC HEATS OF AIR.

By William Thomson.*

I. *Synthetical Investigation of the Duty of a Perfect Thermo-Dynamic Engine founded on the Expansions and Condensations of a Fluid, for which the gaseous laws hold and the ratio* (k) *of the specific heat under constant pressure to the specific heat in constant volume is constant; and modification of the result by the assumption of* Mayer's *hypothesis.*†

Let the source from which the heat is supplied be at the temperature S, and let T denote the temperature of the coldest body that can be obtained as a refrigerator. A cycle of the following four operations, *being reversible in every respect*, gives, according to Carnot's principle, first demonstrated for the Dynamical Theory by Clausius, the greatest possible statical mechanical effect that can be obtained in these circumstances from a quantity of heat supplied from the source.

(1) Let a quantity of air contained in a cylinder and piston, at the temperature S, be allowed to expand to any extent, and let heat be supplied to it to keep its temperature constantly S.

(2) Let the air expand farther, without being allowed to take heat from or to part with heat to surrounding matter, until its temperature sinks to T.

(3) Let the air be allowed to part with heat so as to keep its temperature constantly T, while it is compressed to such an extent that at the end of the fourth operation the temperature may be S.

(4) Let the air be farther compressed, and prevented from either gaining or parting with heat, till the piston reaches its primitive position.

The amount of mechanical effect gained on the whole of this cycle of operations will be the excess of the mechanical effect obtained by the first and second above the work spent in the third and fourth. Now if P and V denote the primitive pressure and volume of the air, and if P_1 and V_1, P_2

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* Extracted from the *Philosophical Transactions* (Part II. 1852), being a Note added to a paper by Mr. Joule on the Air Engine.

† That is, that the heat evolved when air is compressed and kept at constant temperature, is the thermal equivalent of the work spent in the compression—(Addition, April 1853.) Experiments recently made by Mr. Joule and myself have shown that this hypothesis is so nearly true for atmospheric air, that it may be used in all calculations in which the deviations from "the gaseous laws" of compression and expansion are not taken into account. See *Philosophical Magazine*, Oct. 1852.

and V_2, P_3 and V_3, P_4 and V_4, denote the pressure and volume respectively, at the ends of the four successive operations, we have by the gaseous laws, and by Poisson's formula and a conclusion from it quoted above,* the following expressions:—

Mechanical effect obtained by the first operation $= PV \log \frac{V_1}{V}$

Mechanical effect obtained by the second operation

$$= P_2 V_2 \cdot \frac{1}{k-1} \cdot \left\{ \left(\frac{V_2}{V_1} \right)^{k-1} - 1 \right\}.$$

Work spent in the third operation

$$= P_3 V_3 \log \frac{V_2}{V_3}.$$

Work spent in the fourth operation

$$= P_3 V_3 \cdot \frac{1}{k-1} \left\{ \left(\frac{V_3}{V_4} \right)^{k-1} - 1 \right\}.$$

Now, according to the gaseous laws, we have

$$P_1 V_1 = PV, \quad P_2 V_2 = P_1 V_1 \frac{1+ET}{1+ES},$$

and $\qquad P_3 V_3 = P_2 V_2$; and, (since $V_4 = V$,) $P_4 = P$.

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* *From a Letter of the Author's to Mr. Joule.*

"To find the work necessary to compress a given mass of air to a given fraction of its volume, when no heat is permitted to leave the air; let P, V, T be the primitive pressure, volume, and temperature, respectively; let p, v, and t be the pressure, volume, and temperature at any instant during the compression; and let P', V', and T' be what they become when the compression is concluded. Then if k denote the ratio of the specific heat of air at constant pressure to the specific heat of air kept in a space of constant volume, and if, as appears to be nearly, if not rigorously true, k be constant for varying temperatures and pressures, we shall have by the investigation in Miller's 'Hydrostatics' (Edit. 1835, p. 22)—

$$\frac{1+Et}{1+ET} = \left(\frac{V}{v} \right)^{k-1}.$$

But
$$\frac{pv}{PV} = \frac{1+Et}{1+ET},$$

therefore
$$pv = PV \left(\frac{V}{v} \right)^{k-1}.$$

Now the work done in compressing the mass from volume v to volume $v - dv$ will be pdv, or by what precedes,

$$PV.V^{k-1} \frac{dv}{v^k}.$$

Hence by the integral calculus we readily find, for the work, W, necessary to compress from V to V',

$$W = PV \cdot \frac{1}{k-1} \left\{ \left(\frac{V}{V'} \right)^{k-1} - 1 \right\}."$$

Also, by Poisson's formula,

$$\left(\frac{V_2}{V_1}\right)^{k-1} = \left(\frac{V_3}{V}\right)^{k-1} = \frac{1 + ES}{1 + ET}.$$

By means of these we perceive that the work spent in the fourth operation is equal to the mechanical effect gained in the second; and we find, for the whole gain of mechanical effect (denoted by M), the expressions

$$M = (PV - P_3 V_3) \log \frac{V_1}{V} = PV \log \frac{V_1}{V} \cdot \frac{E(S-T)}{1 + ES}.$$

All the preceding formulæ are founded on the assumption of the gaseous laws and the constancy of the ratio (k) of the specific heat under constant pressure to the specific heat in constant volume, for the air contained in the cylinder and piston, and involve no other hypothesis.* If now we add the assumption of Mayer's hypothesis, which for the actual circumstances is $PV \log \frac{V_1}{V} = JH$, H denoting the heat abstracted by the air from the surrounding matter in the first operation, and J the mechanical equivalent of a thermal unit, we have

$$M = JH \cdot \frac{E(S-T)}{1 + ES}.$$

The investigation of this formula given in my paper on the Dynamical Theory of Heat, shews that it would be true for every perfect thermo-dynamic engine, if Mayer's hypothesis were true for a fluid subject to the gaseous laws of pressure and density, whether, for such a fluid (did it exist), k were constant or not.

It was first obtained by using, in the formula

$$M = JH\varepsilon^{-\frac{1}{J}\int_T^S \mu dt},$$

* From the sole hypothesis that k is constant for one fluid fulfilling the gaseous laws and having E for its coefficient of expansion, I find it follows, as a necessary consequence, that Carnot's function would have the form $\frac{JE}{1 + Et + C}$; where C denotes an unknown absolute constant, and t the temperature measured by a thermometer founded on the equable expansions of that gas. From this it follows, that for such a gas subjected to the four operations described in the text, we must have

$$PV \log \frac{V_1}{V} = JH \frac{1 + ES}{1 + ES + C}, \text{ and consequently, } M = JH \frac{E(S-T)}{1 + ES + C}$$

which is Mr. Rankine's general formula.

which involves no hypothesis, the expression

$$\mu = \frac{J}{\frac{1}{E} + t}$$

for Carnot's function, which Mr. Joule had suggested to me, in a letter dated December 9, 1848, as the expression of Mayer's hypothesis, in terms of the notation of my "Account of Carnot's Theory."* Mr. Rankine† has arrived at a formula agreeing with it (with the exception of a constant term in the denominator, which, as its value is unknown, but probably small, he neglects in the actual use of the formula), as a consequence of the fundamental principles assumed in his Theory of Molecular Vortices, when applied to any fluid whatever, subjected to a cycle of four operations satisfying Carnot's criterion of reversibility (being in fact precisely analogous to those described above, and originally invented by Carnot); and he thus establishes Carnot's law as a consequence of the equations of the mutual conversion of heat and expansive power, which had been given in the first section of his paper on the Mechanical Action of Heat.‡

II. *Note on the Specific Heats of Air.*

Let N be the specific heat of unity of weight of any fluid at the temperature t, kept within constant volume, v; and let kN be the specific heat of the same fluid mass, under constant pressure, p. Without any other assumption than than that of Carnot's principle, the following equation is demonstrated in my paper‖ on the "Dynamical Theory of Heat," § 48,

$$kN - N = \frac{\left(\frac{dp}{dt}\right)^2}{\mu \times -\frac{dp}{dv}},$$

where μ denotes the value of Carnot's function, for the temperature t, and the differentiations indicated are with reference to v and t considered as independent variables, of

* Royal Society of Edinburgh, Jan. 2, 1849, *Transactions*, vol. xvi. pt. 5.

† On the Economy of Heat in Expansive Engines. Royal Society of Edinburgh, April 21, 1851, *Transactions*, vol. xx. part 2.

‡ Royal Society of Edinburgh, Feb. 4, 1850, *Transactions*, vol. xx. pt. 1.

‖ Royal Society of Edinburgh, Mar. 17, 1851, *Transactions*, vol. xx. pt. 2.

which p is a function. If the fluid be subject to Boyle's and Mariotte's law of compression, we have

$$\frac{dp}{dv} = -\frac{p}{v};$$

and if it be subject also to Gay-Lussac's law of expansion,

$$\frac{dp}{dt} = \frac{p}{1 + Et}.$$

Hence, for such a fluid,

$$kN - N = \frac{E^2 pv}{\mu(1 + Et)^2}.^*$$

In the case of dry air these laws are fulfilled to a very high degree of approximation, and, for it, according to Regnault's observations,

$$\frac{pv}{1 + Et} = 26215, \quad E = \cdot 00366,$$

(a British foot being the unit of length, and the weight of a British pound at Paris, the unit of force).

We have consequently, for dry air,

$$kN - N = \frac{26215E^2}{\mu(1 + Et)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now it is demonstrated, without any other assumption than that of Carnot's principle, in my "Account of Carnot's Theory" (Appendix III.), that

$$\frac{E}{\mu(1 + Et)} = \frac{H}{W},$$

if W denote the quantity of work that must be spent in compressing a fluid subject to the gaseous laws, to produce H units of heat when its temperature is kept at t. Hence

$$kN - N = 26215E \times \frac{H}{W} = 95 \cdot 947 \times \frac{H}{W} \ldots\ldots (2).$$

If we adopt the values of μ shown in Table I. of the "Account of Carnot's Theory," depending on no uncertain data except the densities of saturated steam at different temperatures, which, for want of accurate experimental data, were derived from the value 1693·5 for the density of

* This equation expresses a proposition first demonstrated by Carnot. See "Account of Carnot's Theory," Appendix III. (*Transactions*, Royal Society of Edinburgh, vol. xvi. part 5.)

saturated vapour at 100°, by the assumption of the "gaseous laws" of variation with temperature and pressure; we find 1357 and 1369 for the values of $\frac{E}{\mu(1+Et)}$ at the temperature 0 and 10° respectively; and hence, for these temperatures,

$$\left.\begin{aligned} (t=0)\quad kN-N&=\frac{95{\cdot}947}{1357}={\cdot}07071 \\ (t=10^\circ)\ kN-N&=\frac{95{\cdot}947}{1369}={\cdot}07008 \end{aligned}\right\}\ldots\ldots\ldots\ldots(a).$$

Or, if we adopt Mayer's hypothesis, according to which $\frac{W}{H}$ is equal to the mechanical equivalent of the thermal unit,* we have $\frac{W}{H}=1390$; and hence, for all temperatures,

$$kN-N=\frac{95.947}{1390}={\cdot}06903\ \ldots\ldots\ldots\ldots(a').$$

The very accurate observations which have been made on the velocity of sound in air, taken in connection with the results of Regnault's observations on its density, &c., lead to the value 1·410 for k, which is probably true in three if not in four of its figures. Now, k being known, the preceding equations enable us to determine the absolute values of the two specific heats (kN, and N) according to the hypotheses used in (a) and (a') respectively; and we thus find,

	Specific heat of air under constant pressure (kN).	Specific heat of air in constant volume (N).
for $t=0$,	·2431	·1724,
for $t=10$,	·2410	·1709,

according to the tabulated values of Carnot's function.

Or, for all temperatures,	·2374	·1684,

according to Mayer's hypothesis.

By the adoption of hypotheses involving that of Mayer, and taking 1389·6 and 1·4 as the values of J and k, respectively, Mr. Rankine finds ·2404 and ·1717 as the values of the two specific heats.

* The number 1390, derived from Mr. Joule's experiments on the friction of fluids, cannot differ by $\frac{1}{100}$, and probably does not differ by $\frac{1}{300}$, of its own value, from the true value of the mechanical equivalent of the thermal unit.

Hence it is probable that the values of the specific heat of air under constant pressure, found by Suermann (·3046), and by De la Roche and Berard (·2669), are both considerably too great; and the true value, to two significant figures, is probably ·24.

Glasgow College, Feb. 19, 1852.

POSTSCRIPT.

In a paper communicated to the Royal Society, along with the above, (March 1852), Mr. Joule described a new experimental determination of the specific heat of air under constant atmospheric pressure, which gave ·23 as a mean result, but he used ·2389 as probably nearer the truth, correcting certain tables, calculated from De la Roche and Berard's result, which he had given in his paper on the Air Engine. M. Regnault has just published (Comtes Rendus, *Ap.* 18, 1853) the results of experimental researches on the specific heat of air, by which he finds that for all temperatures from $-30°$ to $+225°$ centigrade, and for all pressures from one up to ten atmospheres, the specific heat of air is from ·237 to ·2379, and thus both pushes to a minuter degree of accuracy the direct confirmation which the theoretical results published by Mr. Rankine and myself first obtained from Mr. Joule's experiments, and justifies Mr. Joule in the number he actually used in his calculations.

Glasgow College, April 1853.

ON THE CALCULUS OF FORMS, OTHERWISE THE THEORY OF INVARIANTS.

By J. J. Sylvester.

[*Continued from* Vol. VII., p. 214.]

Section VII.

On Combinants.

Reasons of convenience have induced me to depart from the plan to which I originally intended to adhere in the development of this theory, and I shall hereafter, from time to time, continue to add sections on such parts of the subject as may chance to be most present to my mind or most urgent upon my attention, without waiting for the exact place which they ought to occupy in a more formal treatise, and without having regard to the separation of the subject into the two several divisions stated at the outset of the first section. The present section will be devoted to a brief and partial exposition of the theory of Combinants,* with a view to the application of this theory to the solution of the problem of throwing the resultant of three general homogeneous quadratic functions under its most simple form, being analogous

* Discovered by the Author of this paper in the winter of 1852.

to that given by Aronhold in the particular case where the three functions are derived from the same cubic, and becoming identical therewith when the coefficients are accommodated to this particular supposition.* I shall confine myself for the present to combinants relating to systems of functions, all of the same degree.

If $\phi_1, \phi_2, \ldots \phi_r$, be homogeneous functions of any number of variables, any invariant or other concomitant of the system which remains unchanged, not only for linear substitutions impressed upon the variables contained within the functions, but also for linear combinations impressed upon the functions themselves, is what I term a Combinant. A Combinant is thus an invariant or other concomitant of a system in its corporate capacity (quâ *system*), being in fact common to the whole family of forms designated by $\lambda_1.\phi_1 + \lambda_2.\phi_2 + \ldots + \lambda_r.\phi_r$, where $\lambda_1, \lambda_2, \ldots \lambda_r$, are arbitrary constants. If the coefficients of $\phi_1, \phi_2, \ldots \phi_r$, be supposed to be written out in (r) lines (the coefficients of corresponding terms occupying the same place in each line), so as to form a rectangular matrix, any combinative invariant will be a function of the determinants corresponding to the several squares of r^2 terms each that can be formed out of such matrix, or, as they may be termed, the *full* determinants belonging to such rectangular matrix. If we call any such combinant K, then, over and above the ordinary partial differential equations which belong to it in its character of an invariant, it will be necessary and sufficient, in order to establish its combinantive character, that K shall be subject to satisfy $(r-1)$ pairs of equations of the form

$$\left(a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc}\ldots\right)K = 0,$$

$$\left(a\frac{d}{da'} + b\frac{d}{db'} + c\frac{d}{dc'}\ldots\right)K = 0,$$

where $a, b, c \ldots$; $a', b', c' \ldots$, are respectively lines in the matrix above referred to.

So any combinantive concomitant will be a function of the full determinants of the matrix formed by the coefficients of the given system of forms and of the variables, and will be

* A similar method will subsequently be applied to the representation of the resultant of two cubic equations as a function of Combinants bearing relations to the quadratic and cubic invariants of a quartic function of x and y, precisely analogous to those which the Combinants that enter into the solution above alluded to bear to the Aronholdian invariants of a cubic function.

subject to satisfy the additional differential equations just above written.

It will readily be understood furthermore, that an invariant or other concomitant may be combinantive in respect to a certain number of forms of a system, and not in respect of other forms therein; or more generally, may be combinantive in respect of each, separately considered, of a series of groups into which a given system may be considered to be subdivided, without being so in respect of the several groups taken collectively.

In the fourth section of my memoir on a Theory of the Conjugate Properties of two rational integral Algebraical Functions, recently presented to the Royal Society of London, the case actually arises of an invariant of a system of three functions, which is combinantive in respect only to two of them.

For greater simplicity, let the attention for the present be kept fixed upon combinants which are such in respect of a single group of functions, all of the same degree in the variables. (It will of course have been perceived that when the system is made up of several groups, there would be nothing gained by limiting the groups to be all of the same degree *inter se;* it is sufficient that all of the same group be of the same degree *per se.*)

All such combinants will admit of an obvious and immediate classification. Let us suppose that a combinant is proposed which is in its lowest terms, that is to say, incapable of being expressed as a rational integral algebraical function of combinants of an inferior order. Such a combinant may, notwithstanding this, admit of being decomposed into non-combinantive invariants of inferior dimensions to its own, and in such event will be termed a *complex* combinant; or it may be indecomposable after this method, in which event it will be termed a *simple* combinant. It will presently be shewn, that the resultant of a system of three quadratic functions is made up of a complex combinant of twelve dimensions, and of the square of a simple combinant of six dimensions, expressible as a biquadratic function of ten non-combinantive invariants, each of three dimensions in the coefficients. There is an obvious mode of generating complex combinants; according to which they admit of being viewed as invariants of invariants. Supposing $\phi_1, \phi_2, \dots \phi_r$, to be the functions of the given system, $\lambda_1.\phi_1 + \lambda_2.\phi_2 + \dots + \lambda_r\phi_r$, may conveniently be termed the conjunctive of the system: if now one or more invariants or other concomitants be taken

of this conjunctive, there results a derivative function or system of functions of the quantities $\lambda_1, \lambda_2, \ldots \lambda_r$, in which every term affecting any power or combination of powers of the (λ) series is necessarily an invariant or concomitant of the given system. If now an invariant or other concomitant be taken of the new system in respect to $\lambda_1, \lambda_2, \ldots \lambda_r$, (the original variables (supposing them to enter) being treated as constants), this secondarily derived invariant will be itself an Invariant, or at all events a Concomitant in respect of the original system, and being unaffected by linear substitutions impressed upon the λ_1 system, is by definition a combinant of such system. A similar method will obviously apply if the original system be made up of various groups; each group will give rise to a conjunctive, and one or more concomitants being taken of this system of conjunctives and treated as in the case first supposed, (the only difference being, that there will on the present supposition be several *unrelated* systems instead of a single system of new variables, *i.e.* several λ systems instead of one only,) the result, when all the λ systems have been *invariantized out* (*i.e.* made to disappear by any process for forming invariants), will be a combinant in respect to each of the groups, severally considered, of the given system of functions.

Here let it be permitted to me to make a momentary digression, in order to be enabled to avoid for the future the inconvenience of using the phrase "invariant or other concomitant," and so to be enabled at one and the same time to simplify the language and to give a more complete unity to the matter of the theory, by shewing how every concomitant may in fact be viewed as a simple invariant, so that the calculus of forms may hereafter admit of being cited, as I propose to cite it, under the name of the Theory of Invariants.

Thus, to begin with the case of *simple* contragredience and congredience, if $\xi, \eta, \zeta \ldots$ are contragredient to $x, y, z \ldots$, any form containing $\xi, \eta, \zeta \ldots$, which is concomitantive to a given form or system of forms S, which contains $x, y, z \ldots$, may be regarded as concomitantive to the system S', made up of S and the superadded *absolute* form $\xi x + \eta y + \zeta z + \ldots$, say ϑ; where $\xi, \eta, \zeta \ldots$ are treated no longer as variables, but as *constants*. In like manner every system of variables contragredient to $x, y, z \ldots$, or to any other system of variables in S, will give rise to a superadded form analogous to ϑ, the totality of which may be termed S_1; and thus the

various systems $\xi, \eta, \zeta \ldots$ will no longer exist as variables in the derived form, but purely as constants. Again, if S contain any system of variables ϕ, ψ, ϑ, &c., contragredient to x, y, z, &c., the system of variables u, v, w, &c., congredient with x, y, z, &c., may be considered as constants belonging to the superadded form $\phi u + \psi v + \vartheta w \ldots$; but if S do not contain any system contragredient to x, y, z, &c., then u, v, w, &c. may be treated as constants belonging to the superadded system of forms $xv - yu$, $yw - zv$, $zu - xw$, &c.; and so in general any concomitant containing any sets of variables in simple relation, whether of cogredience or contragredience, with any of the sets in the given system S, may in all cases be treated (record such sets) as an *invariant* of the system S', made up of S and a certain superadded system S_1, all the forms contained in which are absolute, by which I mean, that they contain no literal coefficient. The same conclusion may be extended to the case of concomitants containing sets of variables in *compound* relation with the sets in the given system of forms S. Thus, suppose $u_1, u_2, \ldots u_n$, to be in compound relation of cogredience with $x^{n-1}, x^{n-2}.y, x^{n-3}.y^2, \ldots y^{n-1}$; $u_1, u_2, \ldots u_n$, may be regarded as constants belonging to the superadded form

$$u_1.y^{n-1} - (n-1)u_2.y^{n-2}.x + (n-1).\frac{n-2}{2}u_3.y^{n-3}.x^2 \mp \&c. \pm u_n.x^{n-1},$$

say Ω. And thus universally we are now enabled to affirm, that a concomitant of whatever nature to a given system of forms, may be reduced to the form of an invariant of a system made up of the given system and a certain other superadded system of absolute forms: without, therefore, abandoning the use of the terms concomitant, cogredience, contragredience, &c., which for many purposes are highly convenient and save much circumlocution, we may regard every concomitant as a disguised invariant, and under the name of the Theory of Invariants comprise the totality of the theory of Concomitance. I have already had occasion to make use of the superadded form Ω in discussing the theory of the Bezoutiant (a quadratic form concomitant to two functions of the same degree in x, y, which plays a most important part in the theory of the relations of their real roots), in the memoir for the Royal Society previously adverted to.

I now return to the question of applying the theory of combinants to the decomposition of the resultant of three general quadratic functions of x, y, z. It will of course be

apparent that every resultant of any system of n functions of the same degree of a single set of (n) variables is a combinantive invariant of the system. This is an immediate and simple corollary to the theorem given by me in this Journal, in May, 1851. Accordingly, in proceeding to analyse the composition of the resultant of three quadratic functions, I may, besides impressing linear combinations upon the variables, impress linear combinations upon the functions themselves, in any way most conducive to simplicity and facility of expression and calculation; and whatever relations shall be proved to exist between the resultant and other combinants for such specific representation, must be universal, and hold good for the functions in their most general form.

(1) The system, by means of linear substitutions impressed upon the variables which enter into the functions, may be made to assume the form

$$x^2 + y^2 + z^2,$$
$$ax^2 + by^2 + cz^2,$$
$$lx^2 + my^2 + nz^2 + 2pyz + 2qzx + 2rxy.$$

(2) By means of linear combinations of the functions themselves the system may evidently be made to take the form

$$(c-a)\,x^2 + (c-b)\,y^2,$$
$$(a-b)\,y^2 + (a-c)\,z^2,$$
$$ky^2 + 2pyz + 2qzx + 2rxy;$$

and finally, by taking suitable multipliers of x, y, z in lieu of x, y, z, it may be made to become

$$\rho\,(x^2-y^2),$$
$$\sigma\,(y^2-z^2),$$
$$y^2 + 2fyz + 2gzx + 2hxy.$$

We have thus reduced the number of constants in the system from eighteen to five; and as it will readily be seen that in any combinant of the system in its reduced form ρ and σ can only enter as factors of the simple quantity, $(\rho\sigma)^i$, for all purposes of comparison of the combinants of the system of like dimensions with one another, ρ and σ might admit of being treated as being each unity, and accordingly, practically speaking, we have only to deal with three in place of eighteen constants, a marvellous simplification, and which makes it obvious, *à priori*, or at least

affords a presumption almost amounting to and capable of being reduced to certainty, that the number of fundamental combinants of the system, of which all the rest must be explicit rational functions, will be exactly four in number; which, for the canonical form hereinbefore written, on making ρ and σ each unity, will correspond to

$$1,\ f^2+g^2+h^2,\ f^2g^2+g^2h^2+h^2f^2,\ fgh,$$

and will be of the 3rd, 6th, 12th, and 9th degrees respectively. The reason why the squares of f, g, h, instead of the simple terms f, g, h, appear in the 2nd and 3rd of these forms is, because, on changing x into $-x$, y into $-y$, or z into $-z$, two of the quantities f, g, h will change their sign, but the forms representing the invariants of even degrees ought to remain absolutely unaltered for such transformations. I shall in the course of the present section set forth the methods for obtaining these four combinants, which, although of the regularly ascending dimensions 3, 6, 9, 12, belong obviously to two different groups, the one of three dimensions forming a class in itself, and the natural order of the three others being that denoted by the sequence 6, 12, and 9, and not that which would be denoted by the sequence 6, 9, 12, the combinant of the ninth degree being properly to be regarded as in some sort an accidentally rational square root of a combinant of 18 dimensions.

Let now

$$\rho(x^2-y^2)=U,$$
$$\sigma(y^2-z^2)=W,$$
$$y^2+2fyz+2gzx+2hxy=V.$$

The resultant will be found by making

$$x=\pm y,$$
$$z=\pm y,$$

when

$$\left.\begin{matrix}x=+y\\z=+y\end{matrix}\right\},\quad W=(1+2f+2g+2h)y^2,$$

$$\left.\begin{matrix}x=+y\\z=-y\end{matrix}\right\},\quad W=(1-2f-2g+2h)y^2,$$

$$\left.\begin{matrix}x=-y\\z=+y\end{matrix}\right\},\quad W=(1+2f-2g-2h)y^2,$$

$$\left.\begin{matrix}x=-y\\z=-y\end{matrix}\right\},\quad W=(1-2f+2g-2h)y^2.$$

Hence the resultant R

$$=\rho^4.\sigma^4.(1+2f+2g+2h)(1-2f-2g+2h)(1+2f-2g-2h)(1-2f+2g-2h)$$

$$=(\rho\sigma)^4\{(1+2h)^2-4(f+g)^2\}\{(1-2h)^2-4(f-g)^2\}$$

$$=(\rho\sigma)^4\{(1+4h^2-4f^2-4g^2)^2-(4h-8fg)^2\}$$

$$=(\rho\sigma)^4[1-8(f^2+g^2+h^2)+16\{(f^4+g^4+h^4)-2(g^2h^2+h^2f^2+f^2g^2)\} + 64fgh].$$

Let now
$$K=\lambda U+\mu V+\nu W,$$
K being what I term a linear conjunctive of U, V, W. The invariant of K, in respect to x, y, z, will be the determinant

$$\begin{vmatrix} \rho\lambda, & h\mu, & g\mu, \\ h\mu, & \mu-\rho\lambda+\sigma\nu, & f\mu, \\ g\mu, & f\mu, & -\sigma\nu, \end{vmatrix}$$

i.e. $$=(2fgh-g^2)\mu^3+\sigma(h^2-g^2)\mu^2\nu-\rho(f^2-g^2)\mu^2\lambda-\rho\sigma\mu\lambda\nu + \rho^2\sigma\lambda^2\nu-\rho\sigma^2\lambda\nu^2;$$

or, multiplying by 6, we may write

$$I_{x,y,z}.K=6d\lambda\mu\nu+3b_3\mu^2\nu+3b_1\mu^2\lambda+3a_3\lambda^2\nu+3c_1\lambda\nu^2+b_2\mu^3,$$

where
$$d=-\rho\sigma, \qquad b_2=12fgh-6g^2,$$
$$b_1=-2\rho(f^2-g^2), \quad b_3=2\sigma(h^2-g^2),$$
$$a_3=\rho^2\sigma, \qquad c_1=-2\rho\sigma^2,$$

the notation being accommodated to that employed by Mr. Salmon in *The Higher Plane Curves*, pp. 182, 184, λ, μ, ν in $I.K$ being correspondent to x, y, z in Mr. Salmon's form. If now we employ Mr. Salmon's expression (p. 184) for the S (the biquadratic Aronholdian of $I.K$), observing that

$$a_2=0,\ c_2=0,\ a_1=0,\ c_3=0,$$

we have the complex combinant

$$S_{\lambda,\mu,\nu}.I_{x,y,z}.K=a^4-2d^2(b_1c_1+a_3b_3)+da_3b_2c_1-a_3c_1b_1b_3+b_1^2c_1^2+a_3^2b_3^2$$

$$=\rho^4\sigma^4\begin{pmatrix}1-8(f^2+h^2-2g^2)+4(12fgh-6g^2)\\ -16(f^2-g^2)(h^2-g^2)+16(f^2-g^2)^2+(h^2-g^2)^2\end{pmatrix}$$

$$=\rho^4\sigma^4\{1-8(f^2+g^2+h^2)+16(f^4+g^4+h^4-h^2g^2-g^2f^2-f^2h^2)+48fgh\}.$$

Hence, calling the resultant R, we have

$$-3R+4.S_{\lambda,\mu,\nu}.I_{x,y,z}.K=1-8(f^2+g^2+h^2)+16(f^4+g^4+h^4) + 32(f^2g^2+g^2h^2+h^2f^2)$$

$$=\{1-4(f^2+g^2+h^2)\}^2=P^2.$$

Let Ω be taken the polar reciprocal to the conjunctive

$$-\lambda U + \mu V + \nu W;$$

and for greater simplicity, as we know, *à priori*, from the fundamental definition of a combinant which (save as to a factor, must remain unaltered by any linear modification impressed upon the functions to which it appertains), that ρ and σ can enter factorially only in any combinant, let ρ and σ be each taken equal to unity in performing the intermediary operations.

$$\text{Then} \quad \Omega = \left\{ \begin{array}{cccc} -\lambda, & h\mu, & g\mu, & \xi, \\ h\mu, & \lambda+\mu+\nu, & f\mu, & \eta, \\ g\mu, & f\mu, & -\nu, & \zeta, \\ \xi, & \eta, & \zeta, & 0, \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \quad \xi^2(\nu^2+\nu\mu+\nu\lambda+f^2\mu^2) \\ \; + \eta^2(-\lambda\nu+g^2\mu^2) \\ \; + \zeta^2(\lambda^2+\lambda\mu+\lambda\nu+h^2\mu^2) \\ - 2\eta\zeta(f\lambda\mu-hg\mu^2) \\ - 2\zeta\xi\{g(\mu\lambda+\mu\nu)+(g-fh)\mu^2\} \\ - 2\xi\eta(h\mu\nu-fg\mu^2) \end{array} \right\}.$$

Upon Ω, which is a quadratic function in respect of each of the two unrelated systems ξ, η, ζ; λ, μ, ν, and also in respect of the coefficients in (U, V, W), we may operate with the commutative symbol

$$\frac{d}{d\xi}, \; \frac{d}{d\eta}, \; \frac{d}{d\zeta};$$

$$\frac{d}{d\xi}, \; \frac{d}{d\eta}, \; \frac{d}{d\zeta};$$

$$\frac{d}{d\lambda}, \; \frac{d}{d\mu}, \; \frac{d}{d\nu};$$

$$\frac{d}{d\lambda}, \; \frac{d}{d\mu}, \; \frac{d}{d\nu};$$

which, for facility of reference, I shall term 8E.

Considering the first line as stationary, we shall obtain for the value of $8E(\Omega)$ 216 commutatives, which may be expressed under the following form:

$$
\begin{array}{c}
\frac{d}{d\xi},\ \frac{d}{d\eta},\ \frac{d}{d\zeta},\\
\frac{d}{d\xi},\ \frac{d}{d\eta},\ \frac{d}{d\zeta},\\
\left[\frac{d^2}{d\lambda^2},\ \frac{d^2}{d\mu^2},\ \frac{d^2}{d\nu^2}\right]
\end{array}
$$

$$
-\left\{\begin{array}{c}
\frac{d}{d\xi},\ \frac{d}{d\eta},\ \frac{d}{d\zeta}\\
\frac{\dot{d}}{d\xi},\ \frac{\dot{d}}{d\eta},\ \frac{\dot{d}}{d\zeta}\\
\left[\frac{d^2}{d\lambda^2},\ \frac{d}{d\mu}\cdot\frac{d}{d\nu},\ \frac{d}{d\mu}\cdot\frac{d}{d\nu}\right]
\end{array}\right\}
$$

$$
-\left\{\begin{array}{c}
\frac{d}{d\xi},\ \frac{d}{d\eta},\ \frac{d}{d\zeta}\\
\frac{\dot{d}}{d\xi},\ \frac{\dot{d}}{d\eta},\ \frac{\dot{d}}{d\zeta}\\
\left[\frac{d}{d\lambda}\cdot\frac{d}{d\nu},\ \frac{d^2}{d\mu^2},\ \frac{d}{d\lambda}\cdot\frac{d}{d\nu}\right]
\end{array}\right\}
$$

$$
-\left\{\begin{array}{c}
\frac{d}{d\xi},\ \frac{d}{d\eta},\ \frac{d}{d\zeta}\\
\frac{\dot{d}}{d\xi},\ \frac{\dot{d}}{d\eta},\ \frac{\dot{d}}{d\zeta}\\
\left[\frac{d}{d\lambda}\cdot\frac{d}{d\mu},\ \frac{d}{d\lambda}\cdot\frac{d}{d\mu},\ \frac{d^2}{d\nu^2}\right]
\end{array}\right\}
$$

$$
2\left\{\begin{array}{c}
\frac{d}{d\xi},\ \frac{d}{d\eta},\ \frac{d}{d\zeta}\\
\frac{d}{d\xi},\ \frac{d}{d\eta},\ \frac{d}{d\zeta}\\
\left[\frac{d}{d\lambda}\cdot\frac{d}{d\mu},\ \frac{d}{d\mu}\cdot\frac{d}{d\nu},\ \frac{d}{d\nu}\cdot\frac{d}{d\lambda}\right]
\end{array}\right\}.
$$

In this expression the first lines may be considered commutable, the second or dotted lines are subject to the usual process of commutation, which makes three of the six permutations positive and six negative; and the third or bracketed lines are subject to the simple process which

makes all the permutations of the same sign. In the three middle groups two of the terms in the final line are always identical; it will therefore be more convenient to introduce the multiplier 2, and then to consider each such line to represent the three distinct permutations, taken singly.

Let now $$\frac{1}{8}\left\{\frac{d^2}{d\xi^2}\,;\,\frac{d^2}{d\eta^2}\,;\,\frac{d^2}{d\zeta^2}\right\}\Omega = (\Omega),$$

$$\frac{1}{8}\left\{\frac{d^2}{d\xi^2}\,;\,\frac{d}{d\eta}\cdot\frac{d}{d\zeta}\,;\,\frac{d}{d\eta}\cdot\frac{d}{d\zeta}\right\}\Omega = (\Omega)',$$

$$\frac{1}{8}\left\{\frac{d}{d\xi}\cdot\frac{d}{d\zeta}\,;\,\frac{d^2}{d\eta^2}\,;\,\frac{d}{d\xi}\cdot\frac{d}{d\zeta}\right\}\Omega = (\Omega)'',$$

$$\frac{1}{8}\left\{\frac{d}{d\eta}\cdot\frac{d}{d\xi}\,;\,\frac{d}{d\eta}\cdot\frac{d}{d\xi}\,;\,\frac{d^2}{d\zeta^2}\right\}\Omega = (\Omega)''',$$

$$\left\{\frac{d}{d\xi}\cdot\frac{d}{d\eta}\,;\,\frac{d}{d\eta}\cdot\frac{d}{d\zeta}\,;\,\frac{d}{d\zeta}\cdot\frac{d}{d\xi}\right\}\Omega = (\Omega)_1.$$

And let $$\left[\frac{d}{d\lambda^2}\,;\,\frac{d^2}{d\mu^2}\,;\,\frac{d}{d\nu^2}\right] = L,$$

$$\left[\frac{d^2}{d\lambda^2}\,;\,\frac{d}{d\mu}\cdot\frac{d}{d\nu}\,;\,\frac{d}{d\mu}\cdot\frac{d}{d\nu}\right] = L',$$

$$\left[\frac{d}{d\lambda}\cdot\frac{d}{d\nu}\,;\,\frac{d^2}{d\mu^2}\,;\,\frac{d}{d\lambda}\cdot\frac{d}{d\nu}\right] = L'',$$

$$\left[\frac{d}{d\lambda}\cdot\frac{d}{d\mu}\,;\,\frac{d}{d\lambda}\cdot\frac{d}{d\mu}\,;\,\frac{d^2}{d\nu^2}\right] = L''',$$

$$\left[\frac{d}{d\lambda}\cdot\frac{d}{d\mu}\,;\,\frac{d}{d\mu}\cdot\frac{d}{d\nu}\,;\,\frac{d}{d\nu}\cdot\frac{d}{d\lambda}\right] = L_1.$$

Then, attending to the convention just previously explained, we shall have

$$\begin{aligned}\mathrm{E}(\Omega) = {} & (L - 2L' - 2L'' - 2L''' + 2L_1)\\ & \times\{(\Omega) - 2(\Omega)' - 2(\Omega)'' - 2(\Omega)''' + 2(\Omega)_1\},\end{aligned}$$

a symbolical product, any term in which such as $L'.\Omega''$ will mean

$$\left\{\begin{matrix}\left[\dfrac{d^2}{d\lambda^2}\,;\,\dfrac{d}{d\mu}\cdot\dfrac{d}{d\nu}\,;\,\dfrac{d}{d\mu}\cdot\dfrac{d}{d\nu}\right]\\ \dfrac{d}{d\xi}\cdot\dfrac{d}{d\zeta}\,;\,\dfrac{d^2}{d\eta^2}\,;\,\dfrac{d}{d\xi}\cdot\dfrac{d}{d\zeta}\end{matrix}\right\}\tfrac{1}{8}\Omega,$$

and a similar interpretation must be extended to each of the 25 partial products; we have then

$$L(\Omega) = 8g^2, \quad -2L'(\Omega) = 0, \quad -2L'''(\Omega) = 0,$$
$$-2L''(\Omega) = -4g^2, \quad 2L_1(\Omega) = -2,$$
$$-2L(\Omega)' = 0, \quad -2L(\Omega)''' = 0,$$
$$4L'(\Omega)' = 0, \quad 4L''(\Omega)''' = 0,$$
$$4L''(\Omega)' = 0, \quad 4L'''(\Omega)''' = 0,$$
$$4L'''(\Omega)' = 8f^2, \quad 4L'(\Omega''') = 8h^2,$$
$$4L'(\Omega)'' = 0, \quad 4L''(\Omega)'' = 0, \quad 4L'''(\Omega)'' = 0,$$
$$-4L_1(\Omega)' = 0, \quad -4L_1(\Omega)''' = 0,$$
$$-4L_1(\Omega)'' = 4g^2;$$

and, finally, the 5 terms comprised in

$$4\mathrm{E}(\Omega)_1 \text{ each} = 0.$$

All the above equations can be easily verified by direct inspection, it being observed that $8(\Omega)$ represents

$$\nu^2 + \lambda\nu + \lambda^2 + f^2\mu^2; \quad -\lambda\nu + g^2\mu^2; \quad \lambda^2 + \lambda\mu + \lambda\nu + h^2\mu^2;$$

that $8(\Omega)'$ represents

$$\nu^2 + \mu\nu + \lambda\nu + f^2\mu^2; \quad f\lambda\mu - hg\mu^2; \quad f\lambda\mu - hg\mu^2;$$

that $8(\Omega)''$ represents

$$-\lambda\nu + g^2\mu^2; \quad g(\mu\lambda + \mu\nu) + (g - fh)\mu^2; \quad g(\mu\lambda + \mu\nu) + (g - fh)\mu^2;$$

that $8(\Omega)'''$ represents

$$\lambda^2 + \mu\lambda + \nu\lambda + h^2\mu^2; \quad h\mu\nu - fg\mu^2; \quad h\mu\nu - fg\mu^2;$$

and that $(\Omega)_1$ represents

$$f\lambda\mu - hg\mu^2; \quad -(\mu\lambda + \mu\nu) + (g - fh)\mu^2; \quad h\mu\nu - fg\mu^2.$$

We have thus

$$\mathrm{E}(\Omega) = 8g^2 - 4g^2 - 2 + 8f^2 + 8h^2 + 4g^2$$
$$= 2\{4f^2 + 4g^2 + 4h^2 - 1\}.$$

Hence $3R = 4S_{\lambda,\mu,\nu} \cdot I_{x,y,z} \cdot K - \frac{1}{4}\{\mathrm{E}.\Omega\}^2$ .... (*A*).

If we restore to U, V, W their general values, and make

$$U = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$
$$V = a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'zy,$$
$$W = a''x^2 + b''y^2 + c''z^2 + 2f''yz + 2g''zx + 2h''xy,$$

and construct the cubic function,

$$\vartheta = (ax + a'y + a''z)(bx + b'y + b''z)(cx + c'y + c''z)$$
$$- (ax+a'y+a''z)(fx+f'y+f''z)^2 - (bx+b'y+b''z) - (gx+g'y+g''z)^2$$
$$- (cx + c'y + c''z)(hx + h'y + h''z)^2$$
$$+ 2(fx + f'y + f''z)\ (gx + g'y + g''z)(hx + h'y + h''z)$$

i.e. $\Sigma(abc - af^2 - bg^2 - ch^2 + 2fgh)\,x^3,$

$$+\Sigma\{a'bc + ab'c + abc' - (a'f^2 + 2aff') - (b'g^2 + 2bgg') - (c'h^2 + 2chh')$$
$$+ 2f'gh + 2fg'h + 2fgh'\}\,x^2y$$
$$+\{a'b''c + a'bc'' + a''b'c + a''bc' + ab'c'' + ab''c' - 2a'ff'' - 2af'f'' - 2a''ff'$$
$$- 2b'gg'' - 2bg'g'' - 2b''gg' - 2c'hh'' - 2ch'h'' - 2c''hh' + 2f'g''h$$
$$+ 2fg'h'' + 2f''gh' + 2f''gh' + 2fg''h'\}\,xyz.$$

$S_{\lambda,\mu,\nu}\,.\,I_{x,y,z}\,.\,K$ in the preceding equation becomes simply the Aronholdian S to ϑ, which may be calculated by Mr. Salmon's formula previously quoted.

Ω may be taken equal to the determinant

$$\begin{array}{llll} ax + a'y + a''z\,; & hx + h'y + h''z\,; & gx + g'y + g''z\,; & \xi, \\ hx + h'y + h''z\,; & bx + b'y + b''z\,; & fx + f'y + f''z\,; & \eta, \\ gx + g'y + g''z\,; & fx + f'y + f''z\,; & cx + c'y + c''z\,; & \zeta, \\ \quad\xi & ;\quad \eta & ;\quad \zeta & ;\ 0. \end{array}$$

And the cubic commutant of this, obtained by affecting it with the commutative operator,

$$\left.\begin{array}{l} \frac{d}{dx};\ \frac{d}{dy};\ \frac{d}{dz} \\ \frac{d}{dx};\ \frac{d}{dy};\ \frac{d}{dz} \\ \frac{d}{d\xi};\ \frac{d}{d\eta};\ \frac{d}{d\zeta} \\ \frac{d}{d\xi};\ \frac{d}{d\eta};\ \frac{d}{d\zeta} \end{array}\right\}$$

will give $48\mathrm{E}\Omega$ if each of the 4 lines of the operator undergoes permutation, or $8\mathrm{E}(\Omega)$, if one of the 4 lines is kept stationary. Thus it falls within the limits of practical possibility to calculate explicitly, by the formula (A), the value of the resultant. I give to the S of ϑ the appellation of the Hebrew letter ש (*shin*), and to the commutant of Ω the

appellation of the Hebrew letter ט (*teth*). These letters are chosen with design; for I shall presently shew that when the 3 given quadratic functions are the differential derivatives of the same cubic function ψ, the ט becomes the Aronholdian T to the cubic function, or, as we may write it, $T.\psi$, and the ש becomes the Aronholdian S of the Hessian thereto, i.e. $S.H.\psi$.

Thus for the first time the true inward constitution of the resultant of three quadratics is brought to light. The methods anteriorly given by me, and the one subsequently added by M. Hesse for finding this resultant adverted to in Section 2, leads, it is true, to the construction of the form, but throws no light upon the essential mode of its composition.

SOLUTIONS OF PROBLEMS.

(Prob. 2, p. 94.)

The solution of Prob. 2, p. 179, is incorrect. The following is the true solution.

Let a be the radius of the generating circle, θ the inclination of a tangent to the given cycloid to the axis of x. Then the equation to that tangent may be easily shewn to be

$$y\cos\theta - x\sin\theta = a(\pi - 2\theta)\cos\theta \ldots\ldots\ldots\ldots (A).$$

The equation to the cycloid being transcendental, the number of solutions of the problem will be infinite.

Let n be any whole number, and write $(2n+1)\frac{1}{2}\pi + \theta$ in place of θ in equation (A), and then eliminate θ, and we shall have an equation of one case of the required locus.

To take the simplest case, consider only the intersections of any tangent of the given cycloid, with the pair of tangents at right angles to it on either side, whose points of contact are the *nearest* to the point of contact of the first tangent. Here $n=0$, and we have to eliminate θ between the equations

$$\left.\begin{aligned} y\cos\theta - x\sin\theta &= a(\pi - 2\theta)\cos\theta \\ y\sin\theta + x\cos\theta &= \quad - 2a\theta\sin\theta \end{aligned}\right\}.$$

Hence $$x = -\pi a \sin\theta\cos\theta,$$

therefore $$\sin 2\theta = -\frac{x}{\frac{1}{2}\pi \, . \, a},$$

$$y = \pi a \cos^2\theta - 2a\theta,$$

$$= \frac{\pi a}{2}(1 + \cos 2\theta) - 2a\theta,$$

$$= \frac{\pi a}{2}\left(A + \sqrt{\left\{1 - \left(\frac{x}{\frac{1}{2}\pi \,.\, a}\right)^2\right\}} + a \sin^{-1} \frac{x}{\frac{1}{2}\pi \,.\, a}\,.$$

If we would have the equation from the vertex, we must write $x - \frac{1}{2}\pi\, a$ in place of x in this equation, and we get, after reduction,

$$y = \sqrt{(2.\tfrac{1}{2}\pi \,.\, ax - x^2)} + a \operatorname{versin}^{-1} \frac{x}{\frac{1}{2}\pi \,.\, a},$$

the equation to a Trochoid.

(Prob. 8, p. 96.)

Let m be the mass of one of the particles, v its velocity, α the angle at which its line of motion is inclined to the diameter through a point at which it strikes the spherical surface. The impulsive force which it exerts in the impact at this point will be $2mv \cos\alpha$. After striking it will move, with an equal velocity, in another straight line equally inclined to the diameter through the point of impact, and it will therefore impinge again at the same angle, with an equal impulsive force; and the same will hold for all the successive impacts. The interval between any two successive impacts, being the time of describing a chord subtending an angle $2(\frac{1}{2}\pi - \alpha)$ at the centre of the sphere, will be equal to $\frac{2a \cos\alpha}{v}$, where a denotes the radius. Hence the sum of the impulsive forces exerted per unit of time by the particle m is

$$\frac{2mv \cos\alpha}{\frac{2a \cos\alpha}{v}}, \quad \text{or} \quad \frac{mv^2}{a}.$$

To find the continuous pressure per unit of surface equivalent to the effect of the impacts of all the particles, we must divide the sum of the impulsive forces exerted by all the particles in the unit of time by the whole area of the bounding surface. We thus obtain

$$\frac{mv^2}{4\pi a^3},$$

which, if V denote the volume of the hollow space, becomes

$$\frac{\frac{1}{3}mv^2}{V},$$

the required expression.

The second proposition in the enunciation may be demonstrated as a consequence, by simply observing that the pressure at any small part of the surface depends at each instant only on the masses and motions of the particles infinitely near it, and must therefore have the same expression when the vis viva of the motions per unit of volume in its neighbourhood is given, whatever be the dimensions or form of the whole bounding surface.

Second Solution.

The following general demonstration establishes the truth of the proposition at once for a space bounded by a surface of any form.

Let ω denote an infinitely small area of the surface, and let m be the sum of the masses of the infinitely numerous particles in a unit of volume which move in directions towards the same parts contained within definite limits infinitely near a certain straight line inclined at an angle θ to the normal through ω, with velocities between definite limits differing infinitely little from a certain amount, v. The sum of the masses of all these particles which strike the surface within the area ω, in the unit of time, will be

$$mv\omega \cos\theta,$$

since mv is the sum of the masses of all those of them which, in a unit of time, pass a unit plane area perpendicular to their directions of motion. Hence the portion due to the motions of these particles, of the whole pressure experienced by the area ω, is

$$2mv\omega \cos\theta \,.\, v\cos\theta, \quad \text{or} \quad 2\omega \cos^2\theta \,.\, mv^2;$$

and, if Σmv^2 denote the vis viva of all the particles which, whatever be their velocities, move in directions within the prescribed limits, we have

$$2\omega \cos^2\theta \Sigma mv^2$$

for the pressure which they produce on ω. Now if we choose any line parallel to a normal through ω, and any plane through this line, as axis and plane of reference, we may take (θ, ϕ) and $(\theta + d\theta, \phi + d\phi)$ to express by the method of polar directional coordinates the limits prescribed for the motions we have been considering: so that parallels drawn to all these lines of motion, through any point in the axis of reference, will cut the surface of a sphere of unit radius described from this point as centre in an infinitely small area

$$\sin\theta \, d\theta \, d\phi.$$

We must therefore have

$$\Sigma mv^2 = q\,\frac{\sin\theta\, d\theta\, d\phi}{4\pi},$$

if q be the entire vis viva of the motions of particles in all directions, through a unit of volume. By using this in the preceding expression, we find

$$\frac{2\omega q}{4\pi}\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\cos^2\theta\sin\theta\, d\theta\, d\phi$$

for the whole pressure experienced by ω, (the limits of integration for θ being 0 to $\frac{1}{2}\pi$, instead of 0 to π, so that only the motions *toward* and not the motions *from* ω, may be included); and, performing the indicated integrations, we obtain finally, for the pressure on ω,

$$\tfrac{1}{3}\omega q,$$

from which we conclude that $\frac{1}{3}q$ is the pressure per unit of surface, as was to be proved.

Chamonix, Aug. 18, 1853.

(Prob. 9, p. 96.)

Let X, Y, Z, &c. be the points in which parallel lines AX, BY, CZ, &c. meet their opposite spaces, and let the areas of these spaces be abc, &c.; and Pa, Pb, &c. the volumes subtended at any point P by the areas a, b, &c.; and $\alpha\beta\gamma$, &c. the product of the quantities a, b, c, &c. by the sines of the angles between these areas and the common direction of the lines AX, BY, CZ, &c.

Now $\quad V = Pa + Pb + Pc + Pd + \ldots\ldots,$

and therefore $\quad 0 = \alpha + \beta + \gamma + \delta + \ldots\ldots\ldots\ldots\ldots;$

therefore

$$\begin{array}{lllll} & \dfrac{Xb}{\beta} = & \dfrac{Xc}{\gamma} = & \dfrac{Xd}{\delta} = \ldots\ldots = \therefore & -\dfrac{V}{\alpha}, \\ \dfrac{Ya}{\alpha} = & & \dfrac{Yc}{\gamma} = & \dfrac{Yd}{\delta} = & -\dfrac{V}{\beta}, \\ \dfrac{Za}{\alpha} = & \dfrac{Zb}{\beta} = & & = \dfrac{Zd}{\delta} \quad = & -\dfrac{V}{\gamma}. \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & = & - \end{array}$$

And from theory of volumes, as represented by determinants,

$$V^r(xyzt) = \begin{vmatrix} Xa & Xb & Xc & Xd \\ Ya & Yb & Yc & Yd \\ Za & Zb & Zc & Zd \\ Ta & Tb & Tc & Td \end{vmatrix} = \begin{vmatrix} 0 & \beta & \gamma & \delta\ldots \\ \alpha & 0 & \gamma & \delta\ldots \\ \alpha & \beta & 0 & \delta\ldots \\ \alpha & \beta & \gamma & 0\ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \times \frac{(-V)^{r+1}}{\alpha\beta\gamma\delta\ldots}$$

$$= (-V)^{r+1} \begin{vmatrix} 0 & 1 & 1 & 1\ldots \\ 1 & 0 & 1 & 1\ldots \\ 1 & 1 & 0 & 1\ldots \\ 1 & 1 & 1 & 0\ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = (-V)^{r+1}(-1)^r.r. = -V^{r+1}.r;$$

therefore $\qquad (xyzt\ldots\ldots) = -rV.$

In the same way, if the lines passed through the point whose coordinate volumes are Va, Vb, &c., we should find that

$$V_1 = (-1)^r.rV.\frac{Va\,Vb\,Vc\ldots}{(V-Va)(V-Vb)(V-Vc)\ldots}$$

(Prob. 10, p. 96.)

LET x_p, y_p be the coordinates of a point P referred to the axes of the conic; then, if t is its polar, 0_t the perpendicular on it from centre,

$$\frac{x_p}{a^2} = \frac{\sin tx}{0_t}, \quad \frac{y_t}{b^2} = \frac{\cos tx}{0_t}.$$

Hence, for any two points a and b,

$$\frac{x_a y_b - x_b y_a}{a^2 b^2} = \frac{\sin 1x \cos 2y - \sin 1y \cos 2x}{0_1 0_2};$$

therefore $\qquad 2(0ab) = \dfrac{\sin(12)}{0_1 0_2}\,a^2b^2;$

$$\therefore 2(0ab + 0bc + 0ca) = 2(abc) = \frac{a^2b^2}{0_1 0_2 0_3}\,\Sigma \sin 12.0_3 = a^2b^2\,\frac{(123_p)}{0_1 0_2 0_3}.$$

Now if p be the perpendicular from the centre on the tangent, r the radius of curvature, $rp^3 = a^2b^2$. Hence, as the triangle formed from two tangents and the chord of contact is its own polar triangle, in the limit the radii will be as 1, 2, 4; and the areas of the inscribed and circumscribed triangles as 1, 2.

In space of r dimensions, if V be the volume formed from the semiaxes of the quadratic surface, we shall have, using

the notation of the last question,

$$0a.0b.0c\ldots(123\ldots) = V^2(abc\ldots)^r.$$

From these expressions, any pure identities of areas, volumes, &c., considered as formed from points, are dualized. Thus, having the area of a Pascal triangle in terms of the areas derived from the original points, we can write down the corresponding expression for the Brianchon triangle. But there is also another form of this function, which in a different point of view is not of less importance; I mean that in which one of the points is considered as variable. Here, if we have given the point equation to a surface in space r, involving any number of ordinates, by certain operations upon it, we can find its line, plane, &c. equations, the last of which (the $(r-1)$ space equation) involves this function. Thus, in space of 2 dimensions, if the equation involves the ordinates p_1 and p_2 referred to lines 1 and 2, its reciprocal equation will involve $(t12_\rho)$. In cases like this, the most convenient form is $\sin 12.(12_t)$, (12_t) denoting the perpendicular from the point 12 on the line t. Thus, the equation

$$Ap_1^2 + Bp_2^2 + Cp_3^2 = 0$$

gives the reciprocal equation

$$\frac{\{\sin 12.(12_t)\}^2}{A} + \frac{\{\sin 23.(23_t)\}^2}{B} + \frac{\{\sin 31.(31_t)\}^2}{C} = 0.$$

But it is very remarkable that this is an invariant form of *all space*, so that the polar reciprocal of any point equation of a surface will involve functions such as

$$\sin(12\ldots r).(12\ldots r_t),$$

where $\sin(12\ldots r)$ denotes that function of the angle formed from the r inferior spaces corresponding to the sine of an angle between 2 lines, and $(12\ldots r_t)$ the perpendicular from their intersection on the space t.

(Prob. 1, p. 188.)

Suppose that the two conics are projected into small circles of the same sphere; converting the theorem, we have then to shew that if two small circles of the same sphere are projected upon a plane (the centre of the sphere being the centre of projection*) the normal distance of the point

* This was intended in the problem proposed.

of intersection of the transverse or major axes from each of the conics is one and the same pure imaginary quantity. This property might be deduced from the known properties of focal conics, but it is easier to derive it directly by a method which indeed contains implicitly the demonstration of these properties of focal conics.

Let the centre of the sphere be taken as the origin, the axes of z being perpendicular to the plane of the conics; then, writing for the sphere, and the plane of one of the small circles,

$$\begin{cases} x^2 + y^2 + z^2 = 1, \\ lx + my + nz = 1, \end{cases}$$

the equation of the projecting cone is

$$x^2 + y^2 + z^2 - (lx + my + nz)^2 = 0.$$

And if in this equation we suppose z equal to the perpendicular distance of the centre from the plane of the conics ($z = \gamma$ suppose), we have

$$x^2 + y^2 + \gamma^2 - (lx + my + n\gamma)^2 = 0,$$

for the equation of one of the conics: this equation shews that the conic must have a double contact with the imaginary circle

$$x^2 + y^2 + \gamma^2 = 0;$$

i.e. the centre of this circle (viz. the foot of the perpendicular let fall from the centre of the sphere upon the plane of the conics) must lie on an axis of the conic. Moreover the radius of the circle, *i.e.* the normal distance of its centre from the conic, is $i\gamma$, or the perpendicular in question multiplied by the imaginary symbol i; and since the circle $x^2 + y^2 + \gamma^2 = 0$ is the same whatever the small circle on the sphere may be, the other conic must satisfy the same conditions, or the foot of the perpendicular must be the point of intersection of the axes of the two conics, and its normal distance from each of the conics must be one and the same pure imaginary quantity. It only remains to shew that the axes must be the transverse or major (or more correctly 'focal') axes; in fact, the normal distance from a conic of any point upon the non-focal axis is in every case—even when the normal itself is imaginary—a real positive quantity, so that it is only in the case where the point is upon the focal axis that the normal distance can be a pure imaginary quantity.

(Prob. 5, p. 188.)

(*a*) Instead of heating the air directly, we can produce the required effect more economically by means of a perfect thermodynamic engine; and it is easy to shew that this is the most economical way. We will consider the air heated pound by pound, and sent into the building at the end of the heating process. Generally, let T be the temperature of the unheated air, S the temperature to which we wish it heated. T being the temperature of air, water, &c. external to the building, will be the temperature of our refrigerator; the pound of air to be heated will be our source (nominally), and by working the engine backwards instead of taking away, we will give heat to the source.

If a be the specific heat of air, adt units will be required to raise the temperature of the pound of air from t to $t+dt$, and the work which must be spent to supply this will be

$$Jadt\,\frac{t-T}{t+\frac{1}{E}}.^{*}$$

Let the whole work spent upon the pound of air be denoted by W; then we have

$$W = Ja\int_T^S \frac{t-T}{t+\frac{1}{E}}\,dt;$$

whence $$W = Ja\left\{(S-T)-\left(T+\frac{1}{E}\right)\log\frac{S+\frac{1}{E}}{T+\frac{1}{E}}\right\}.$$

Ex. $S = 80°$, $T = 50°$, Fahrenheit.

As E is usually given with reference to units centigrade, we prefer reducing to that scale.

$$S = 26°\ 66' \text{ and } T = 10° \text{ Cent.}, \quad E = {\cdot}00366.$$

$$\frac{1}{E} = 273{\cdot}2240437, \quad a = {\cdot}24, \quad J = 1390.$$

* For the formulas regarding the duty of a perfect engine, and the mechanical value of each of its cycle of operations, constantly to be employed in these solutions, we refer to a paper by Professor W. Thomson in the *Philosophical Transactions*, and which appears in the present number of this *Journal*.

$$W = 1390 \times {\cdot}24 \left\{16{\cdot}66' - 283{\cdot}22404 \log \frac{299{\cdot}89071}{283{\cdot}22404}\right\}$$

$$= 1390 \times {\cdot}24 \{16{\cdot}66' - 16{\cdot}194713\}$$

$$= 157{\cdot}4438.$$

As one pound of air is heated per second, the H. P. of the engine will be got by dividing this by 550, so that

H. P. of engine = ·28626.

If an engine (probably a steam-engine) be employed to drive the heating machine, and economise only $\frac{1}{10}$th of the fuel, the fuel must have evolved $10\frac{W}{J}$ units. To heat the pound directly $a(S - T)$ units must be supplied, and $\dfrac{10\frac{W}{J}}{a(S-T)} \times 100$ gives the per-centage. In the particular case we have been considering, we find that to heat the air by means of an engine economising $\frac{1}{10}$, would require $\dfrac{{\cdot}47195}{16{\cdot}6'} \times 100$, or 28·317 per cent. of the fuel required for direct heating.

(*b*) Conceive two double-stroke cylinders connected by tubes and valves in some convenient way, with a reservoir between them. Conceive the one to be made of perfectly conducting matter, so that there shall be no difference in temperature between internal and external air, (practically this may be approximated to by immersion in running water); the other cylinder must, on the contrary, be perfectly non-conducting. The pressure in the reservoir being kept at an amount depending on the required heating effect, air is admitted (doing work as it enters) by the former, which we shall call the ingress cylinder, and is not allowed to cool below atmospheric temperature. It is pumped out by the latter, called the egress cylinder, and so heated by compression to the required temperature.

After these very general explanations, we proceed to mention more particularly the *details* of this process.

Let p', t' be respectively the atmospheric pressure and temperature, v' the volume of one pound of air under

pressure p' and at temperature t', t the temperature to which we wish the air to be raised, p the pressure such, that, if air under it be compressed to pressure p', the temperature will rise from t' to t, v the volume of one pound of air under pressure p and at temperature t', v_1 the volume of air under pressure p' and at temperature t. Now, by Poisson's formula and by the gaseous laws we have

$$\text{(A)} \qquad \frac{\frac{1}{E} + t'}{\frac{1}{E} + t} = \left(\frac{p}{p'}\right)^{\frac{K-1}{K}}, \quad p = p'\left\{\frac{\frac{1}{E} + t'}{\frac{1}{E} + t}\right\}^{\frac{K}{K-1}},$$

from which p may be determined. p must be, moreover, the pressure in the reservoir, as will afterwards appear.

Volume of Cylinder. As the apparatus has to supply one pound of air per second, it will be convenient to suppose the cylinders of such a size, as to contain one pound of air at pressure p and temperature t'.

Operations in Ingress Cylinder. Suppose the piston at the top of its stroke, and the lower part of the cylinder connected with the reservoir, and consequently filled with air at pressure p and temperature t'. Then external air admitted above the piston will push it down ($p' > p$). In this the first part of the stroke, admit so much air that, when secondly it is allowed to expand at constant temperature t', we will have reached the end of the stroke by the time that the pressure has fallen to p. The lower part of the cylinder having been connected with the reservoir, has given to the latter the pound of air it contained; and at the end of the down-stroke the upper part is filled with air ready to be sent in by the up-stroke.

In this operation it is plain that we obtain mechanical effect, and we will naturally spend it, in helping to pump the air out by the egress cylinder.

Operations in Egress Cylinder. Suppose, as before, the piston at the top of its stroke, and the cylinder filled with air at pressure p and temperature t'. During the whole stroke you allow air from the reservoir to enter above the piston. During the first part of the stroke you compress the air below the piston, until the pressure becomes p' and the temperature consequently t. Then expel this heated air into the building or whatever place you wish to heat.

Estimate of total work spent.

(1) In egress cylinder:
mechanical effect obtained during the first part of the stroke

$$= (p' - p)\, v';$$

mechanical effect obtained during the second part

$$= p'v' \log \frac{v}{v'} - p(v - v').$$

(B) Whole gain in ingress cylinder

$$= p'v' \log \frac{v}{v'}.$$

(2) In egress cylinder:
mechanical effect spent during compression

$$= \frac{pv}{K-1} \left\{ \left(\frac{v}{v} \right)^{K-1} - 1 \right\} - p(v - v_1);$$

work spent during expulsion

$$= p'v_1 - pv_1:$$

but
$$\frac{p'v_1}{pv} = \frac{\frac{1}{E} + t}{\frac{1}{E} + t'} = \left(\frac{v}{v_1} \right)^{K-1}.$$

(C) Whence whole work spent in egress cylinder

$$= \frac{pv}{K-1} \left\{ \frac{\frac{1}{E} + t}{\frac{1}{E} + t'} - 1 \right\} + pv \left\{ \frac{\frac{1}{E} + t}{\frac{1}{E} + t'} \right\} - pv,$$

$$= p'v' \frac{K}{K-1} \frac{t - t'}{\frac{1}{E} + t'}, \quad \text{since } pv = p'v'.$$

(D) Amount of work spent in both

$$= p'v' \frac{K}{K-1} \frac{t - t'}{\frac{1}{E} + t'} - p'v' \log \frac{v}{v'}.*$$

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* Modifying this by means of the formulas

$$\frac{v}{v'} = \left\{ \frac{\frac{1}{E} + t}{\frac{1}{E} + t'} \right\}^{\frac{K}{K-1}} \quad \text{and } (E),$$

Ratios of expansion:

in the first cylinder, $\frac{v'}{v} = \frac{p}{p'} = \left\{\frac{\frac{1}{E} + t'}{\frac{1}{E} + t}\right\}^{\frac{K}{K-1}}$,

in the second cylinder, $\frac{v_1}{v} = \left\{\frac{\frac{1}{E} + t'}{\frac{1}{E} + t}\right\}^{\frac{1}{K-1}}$.

A slight consideration will shew, that the rates of the cylinders must be the same if we consider them as of the same size, and as each contains one pound of air at pressure p and temperature t', the rate will evidently be 30 double strokes per minute.

Let h be the height of the cylinder, and r the radius of the base; then volume of cylinder $= v = \pi r^2 h$, and if V_0 be the volume of a pound of air under pressure p' and at 0° Cent.,

(E) $$v' = V_0(1 + Et'),$$

$$v = \frac{p'}{p} V_0(1 + Et').$$

The preceding formulas give us the means of calculating readily the most useful results.

We will take as an example, to supply a building with one pound of air heated mechanically from 50° to 80° Fahr. (solved before, in question (a)). We have then $t' = 10°$ Cent., $t = 26\cdot6'$ Cent., $E = \cdot00366$, $\frac{1}{E} = 273\cdot22404$, $p' = 2114$

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we find the following as equivalent:

$$\frac{K}{K-1} Ep'V_0 \left\{(t - t') - \left(t' + \frac{1}{E}\right) \log \frac{t + \frac{1}{E}}{t' + \frac{1}{E}}\right\};$$

which becomes identical with the expression given in division (a) when we substitute for $\frac{K}{K-1} Ep'V_0$ the value Ja, which it must have in consequence of the relation between the specific heats of air and the mechanical equivalent of the thermal unit established in another paper in this number of the *Journal*.

pounds per square foot. $K = 1{\cdot}41$. Then, by (A),

$$p = 2114\left(\frac{283{\cdot}22404}{299{\cdot}89071}\right)^{\frac{1{\cdot}41}{{\cdot}41}},$$

$$= 1736{\cdot}6189 = 2114 \times {\cdot}8214848,$$

$$= 2114 \times \frac{1}{1{\cdot}217308}.$$

All these forms are useful.

Volume of cylinder. $v' = V_0(1 + 10E)$,

$$= 12{\cdot}383 \times 1{\cdot}0366,$$

$$= 12{\cdot}836218.$$

$$pv = p'v'. \quad v = \frac{p'}{p}v',$$

$$= 1{\cdot}217308 \times 12{\cdot}836218.$$

Volume of cylinder $= v = 15{\cdot}6256$.

A practically useful height of cylinder might be 4 feet, the corresponding diameter to which is 2·2302 feet.

Again, we have, amount of work spent in cooling in this way one pound of air (D)

$$= \frac{K}{K-1}p'v'\left\{\frac{\frac{1}{E}+t}{\frac{1}{E}+t'} - 1\right\} - p'v'\log\frac{v}{v'}$$

$$= \frac{141}{41} \times 2114 \times 12{\cdot}836\left\{\frac{299.89071}{283{\cdot}22404} - 1\right\}$$

$$- 2114 \times 12{\cdot}836 \log 1{\cdot}217308$$

$$= 5491{\cdot}54 - 5336{\cdot}03$$

$$= 155{\cdot}51 \text{ estimated in foot pounds.}$$

As one pound must be supplied per second, H. P. of engine required to drive the apparatus = ·2827. This result ought, inasmuch as this apparatus possesses all the qualifications of a perfect engine, to be identical with the answer found in division (*a*) of this problem; we however find a difference of ·0034 between the two, due to the circumstance that the number ·24 which we employed as the value of the specific heat of air in the previous solution, also 1·41 for K in this, are only approximately true; but the true H. P. to two significant figures is ·28.

Ratios of Expansion. In first cylinder

$$\frac{v'}{v} = \cdot 8214848,$$

so that 3·2859 feet of the stroke passed while air was being admitted at pressure 2114, and ·71406 feet in allowing this to expand to pressure of receiver.

In second cylinder

$$\frac{v_1}{v_1'} = \cdot 869825,$$

or ·5207 feet of the stroke was spent in compressing the air from pressure p to pressure p' or 2114, the remaining 3·4793 feet in expelling it.

(*c*) The first suggestion, we believe, of an apparatus for cooling buildings by compressing air, was to pump in air into a reservoir and allow it to cool to the temperature of the atmosphere, on the supposition that if then allowed to rush out by means of a stopcock, it would, in consequence of the expansion, fall in temperature. Unfortunately however for this scheme, it has been found that there is only an almost imperceptible depression of temperature (after motion ceases in the air) due to a want of perfect rigour in Mayer's hypothesis. The friction of the air in the orifice &c. almost entirely compensates for the cold of expansion.

The apparatus described in (*b*) can, however, be very simply applied.

Instead of allowing the air to rush out, and thus heat itself by friction, let it out slowly, and make it work a piston in a double-stroke cylinder, and we shall not only obtain the full benefit of the cold of expansion, but also gain so much work as to make the H. P. of the engine required to drive the apparatus a mere trifle.

The working of the apparatus, however, will not be so simple as in the last case, for as we are to use the same apparatus, we cannot make the cylinders hold one pound of air, and cannot even have the pistons moving at the same rate.

Let p' and t' be the atmospheric pressure and temperature respectively, v' the volume of either cylinder, t the temperature of the cooled air, p the pressure in the receiver, which will be such that if air at pressure p and temperature t' be allowed to expand to pressure p' the temperature will become t, v the volume under pressure p of a quantity of air,

which under pressure p' would fill the cylinder, there being no change of temperature, v_1 volume under pressure p and temperature t', of a quantity of air which would fill the cylinder under pressure p' and at temperature t.

$$\text{(A}'\text{)} \qquad \frac{p}{p'} = \left\{\frac{\frac{1}{E} + t'}{\frac{1}{E} + t}\right\}^{\frac{K}{K-1}}.$$

Operations in Ingress Cylinder. Suppose the piston at the top of its stroke, the cylinder full of air at ordinary pressure. Admitting external air above the piston, push the piston down until the air below is compressed to pressure p, the temperature being kept constant; and then send this compressed air into the reservoir.

Operations in Egress Cylinder. In the first part of the stroke allow so much air to enter the cylinder from the reservoir, that if allowed to expand in the remaining part of the stroke, the pressure and temperature at the end will be p' and t respectively.

In Ingress Cylinder. Work spent in compressing air from p' to p (temperature constant)

$$= p'v' \log \frac{v'}{v} - p'(v' - v).$$

Work spent in sending the air into the receiver

$$= pv - p'v.$$

(B′) Total expenditure per stroke in first cylinder

$$= p'v' \log \frac{v'}{v}.$$

In Egress Cylinder. Mechanical effect gained in partially filling the cylinder from reservoir

$$= pv_1 - p'v_1.$$

Mechanical effect gained during the rest of the stroke

$$= \frac{p'v'}{K-1}\left\{\left(\frac{v'}{v_1}\right)^{K-1} - 1\right\} - p'(v' - v_1).$$

(C′) Total gain per single stroke

$$= \frac{p'v'}{K-1}\left\{\left(\frac{v'}{v_1}\right)^{K-1} - 1\right\} + pv_1 - p'v'$$

$$= \frac{p'v'}{K-1}\left\{\frac{\frac{1}{E}+t'}{\frac{1}{E}+t} - 1\right\} + p'v'\left\{\frac{\frac{1}{E}+t'}{\frac{1}{E}+t} - 1\right\}$$

$$= p'v' \frac{K}{K-1} \frac{t'-t}{t+\frac{1}{E}}.$$

Ratios of Expansion. In ingress cylinder,

$$\frac{v}{v'} = \frac{p'}{p} = \left\{\frac{\frac{1}{E}+t}{\frac{1}{E}+t'}\right\}^{\frac{K}{K-1}}.$$

In egress cylinder,

$$\frac{v_1}{v'} = \left\{\frac{\frac{1}{E}+t}{\frac{1}{E}+t'}\right\}^{\frac{1}{K-1}}.$$

It would be a matter of no difficulty to give generally the number of pounds each cylinder would contain, and also the rate of each piston, but it would only tend to confusion of symbols, as we should have to take in the p's, t's, and v's of (b), as well as those of this case; we will give these details in the example, which is

$t' = 80°$ Fahrenheit $= 26{\cdot}6'$ Cent., $t = 50°$ Fahrenheit $= 10$ Cent., p' of course 2114, height of cylinder, as before, 4 feet, and consequently, diameter 2·2302 feet.

From (A′) we have $p = 2114 \times 1{\cdot}217308$ pounds per square foot.

In (b) we found volume of cylinder $= 15{\cdot}6256 = v'$, according to our present notation.

From (B′) we have total expenditure per single stroke in ingress cylinder

$$= p'v' \log\frac{v'}{v} = p'v' \log\frac{p}{p'}$$

$$= 1{\cdot}217308 \times \{\text{the gain in ingress cylinder in } (b)\}$$

$$= 6495{\cdot}59.$$

From (C′) we have total gain per single stroke in egress cylinder

$$= 1{\cdot}217308 \times \{\text{loss in egress cylinder in } (b)\}$$

$$= 6684{\cdot}9.$$

Ratios of Expansion. In ingress cylinder

$$\frac{v}{v'} = \cdot 8214848,$$

and in egress

$$\frac{v_1}{v'} = \cdot 869825,$$

as in the example attached to (b).

Let χ be the number of pounds the first cylinder contains at pressure p' (2114), and at temperature t' (26·6′ Cent.).

Volume of χ pounds $= 12 \cdot 383\, \chi\, (1 + E \times 26 \cdot 6) =$ volume of cylinder $= 12 \cdot 383\, (1 + E \times 10) \times 1 \cdot 217308$, as we found in ($b$); hence

$$\chi = \frac{283 \cdot 22404}{299 \cdot 89071}\, 1 \cdot 217308 = 1 \cdot 149655.$$

Number of single strokes per second

$$= \frac{1}{\chi} = \cdot 8698348.$$

Number of double strokes per minute

$$= 26 \cdot 095044.$$

In the second cylinder, let χ' be the number of pounds contained by it at p' (2114) and t (10° Cent.), we easily obtain

$$\chi' = 1 \cdot 217308.$$

Number of strokes per second

$$= \frac{1}{\chi'} = \cdot 8214848.$$

Number of double strokes per minute

$$= 24 \cdot 644544.$$

Whole work spent per second in ingress cylinder

$$= \frac{6495 \cdot 59}{\chi} = 5650 \cdot 036.$$

Whole mechanical effect gained per second in egress cylinder

$$= \frac{6684 \cdot 9}{\chi'} = 5491 \cdot 54.$$

Total motive power required $= 5650 \cdot 036 - 5491 \cdot 54$

$= 158 \cdot 496$ ft. pounds per second.

Hence, H.P. of engine required to drive the apparatus $= \cdot 288$.

(Prob. 7, p. 189.)

Let Δ = area of triangle, $rc = \sin C$,

$$\frac{du}{d\alpha} = u', \quad \frac{du}{d\beta} = v', \quad \frac{du}{d\gamma} = w',$$

and let δ be the distance between two points $(\alpha\beta\gamma)$, $(\alpha_1\beta_1\gamma_1)$; and $(\alpha_2\beta_2\gamma_2)$ the differences of their coordinates. Then it is easy to shew that

$$2\Delta r\delta^2 + a\beta_2\gamma_2 + b\gamma_2\alpha_2 + c\alpha_2\beta_2 = 0 \text{ .... } (1).$$

Let the equation to the inscribed ellipse be

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2ln\gamma\alpha - 2lm\alpha\beta = 0 \text{...} (2).$$

Let $(\alpha_1\beta_1\gamma_1)$ be the centre; then δ is a semidiameter, and the equations to the centre are

$$\frac{u_1'}{a} = \frac{v_1'}{b} = \frac{w_1'}{c} = k \text{ .... } (3),$$

or $$\frac{a\alpha_1}{mb^{-1} + nc^{-1}} = \frac{b\beta_1}{nc^{-1} + la^{-1}} = \frac{c\gamma_1}{la^{-1} + mb^{-1}} = -k\frac{abc}{lmn} \text{ ... } (4).$$

But $$a\alpha_1 + b\beta_1 + c\gamma_1 = 2\Delta \text{ identically .... } (5),$$

therefore $$a\alpha_2 + b\beta_2 + c\gamma_2 = 0 \text{ .... } (6),$$

and if $$la^{-1} + mb^{-1} + nc^{-1} = \lambda \text{ .... } (7),$$

by (4), (5), and (7), $$k\lambda + lmn = 0 \text{ .... } (8).$$

By (3) and (5), for all values of α, β, γ,

$$\alpha u_1' + \beta v_1' + \gamma w_1' = 2\Delta k \text{ .... } (9),$$

therefore $$u_1 = \Delta k \text{ .... } (10).$$

By Taylor's theorem,

$$u_2 = u_1 - (\alpha u_1' + \beta v_1' + \gamma w_1') + u \text{......}$$

By (2), (9), (10), $$= -k\Delta \text{ .... } (11).$$

If δ be a semiaxis, δ^2 is a maximum or minimum; therefore if μ_1, μ_2 are arbitrary, we have, from (1), (6), and (11),

$$a\mu_1 + u_2'\mu_2 + b\gamma_2 + c\beta_2 = 0 \text{ .... } (12),$$

$$b\mu_1 + v_2'\mu_2 + c\alpha_2 + a\gamma_2 = 0 \text{ .... } (13),$$

$$c\mu_1 + w_2'\mu_2 + a\beta_2 + b\alpha_2 = 0 \text{ .... } (14);$$

therefore, by (1), (6), and (11),

$$k\mu_2 + 2r\delta^2 = \alpha_2(12) + \beta_2(13) + \gamma_2(14) = 0 \text{ ... } (15).$$

The result of substituting in (6) proportionals to α_2, β_2, γ_2,

found from (12), (13), and (14), is a quadratic in μ_2 or δ^2, and the theory of cross-multiplication shews that the absolute term does not contain l, m, or n, and that the coefficient of μ_2

$$= 4a(2lmn^2b + 2lm^2nc) + \&c.,$$

and therefore, by (7), $\qquad \propto lmn\lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16).$

But $\qquad$ (area of ellipse)$^2 \propto$ product of values of δ^2;

by (15) and (16), $\qquad \propto (k^2lmn\lambda)^{-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (17).$

This area is to be a maximum; therefore if $\rho_1\rho_2$ are arbitrary, we have from (7), (8), and (17), writing down the coefficients of dl and dm only,

$$\rho_1 l^{-1} + \rho_2 l^{-1} + a^{-1} = 0,$$

$$\rho_1 m^{-1} + \rho_2 m^{-1} + b^{-1} = 0;$$

therefore $\qquad la^{-1} = mb^{-1} = nc^{-1},$

and the equation to the curve becomes

$$(a\alpha)^{\frac{1}{2}} + (b\beta)^{\frac{1}{2}} + (c\gamma)^{\frac{1}{2}} = 0.$$

The equation to the minimum circumscribed ellipse may be established by precisely similar reasoning.

NOTE 1.—If the curve (2) be an inscribed circle,

$$\alpha_1 = \beta_1 = \gamma_1;$$

then, by (4),

$$a^{-1}\left(\frac{m}{b} + \frac{n}{c}\right) = b^{-1}\left(\frac{n}{c} + \frac{l}{a}\right) = c^{-1}\left(\frac{l}{a} + \frac{m}{b}\right),$$

and if $2s = a + b + c$,

$$\frac{l}{a(s-a)} = \frac{m}{b(s-b)} = \frac{n}{c(s-c)},$$

or $$\frac{l}{\cos^2\frac{1}{2}A} = \frac{m}{\cos^2\frac{1}{2}B} = \frac{n}{\cos^2\frac{1}{2}C},$$

and the equation is

$$\alpha^{\frac{1}{2}}\cos\tfrac{1}{2}A + \beta^{\frac{1}{2}}\cos\tfrac{1}{2}B + \gamma^{\frac{1}{2}}\cos\tfrac{1}{2}C = 0.$$

If the circle touch AB and AC produced, α is negative, and the equation is

$$0 = (-\alpha)^{\frac{1}{2}}\cos\tfrac{1}{2}A + \beta^{\frac{1}{2}}\sin\tfrac{1}{2}B + \gamma^{\frac{1}{2}}\sin\tfrac{1}{2}C.$$

NOTE 2.—If p_1, p_2, p_3 be the perpendiculars from A, B, C on any straight line, its equation will be

$$a\alpha p_1 + b\beta p_2 + c\gamma p_3 = 0.$$

NOTE ON PRECEDING SOLUTION.

[The equations to the maximum and minimum ellipse respectively inscribed in, and circumscribed about, a given triangle, may also be obtained very simply from the consideration, that if the triangle be projected so as to become equilateral, the ellipse must become a circle. Now it is a property of an equilateral triangle inscribed in a circle, that the tangents at its angles are parallel to the sides respectively opposite to them. This property will not be affected by projection. But if the equation to the circumscribed ellipse be

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0,$$

the equation to the tangent at $(\beta\gamma)$ will be

$$n\beta + m\gamma = 0;$$

and in order that this may be parallel to $\alpha = 0$, it is necessary that we have

$$\frac{n}{b} = \frac{m}{c},$$

or $$mb = nc = (\text{by symmetry}) = la.$$

Hence the equation to the minimum circumscribed ellipse becomes

$$(a\alpha)^{-1} + (b\beta)^{-1} + (c\gamma)^{-1} = 0.$$

That the maximum inscribed ellipse will be obtained in a precisely similar way.]

END OF VOLUME VIII.

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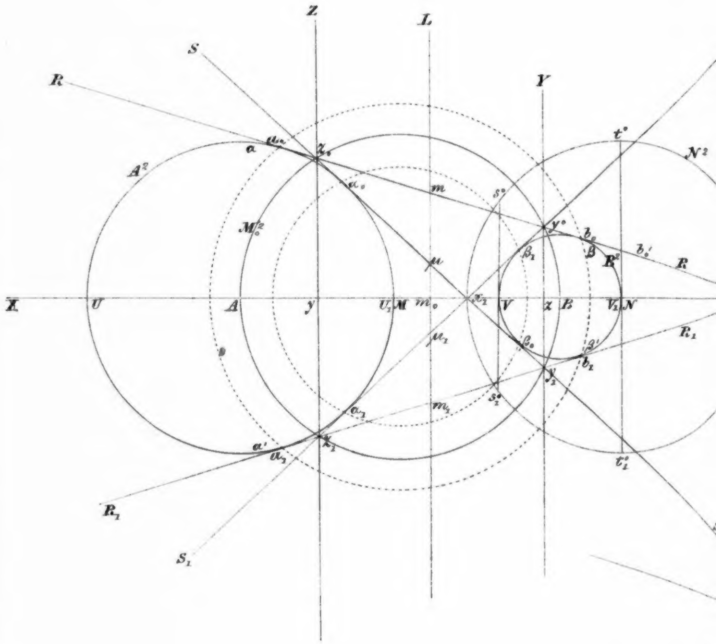
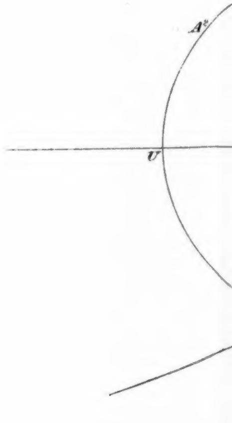
Fig. 1.





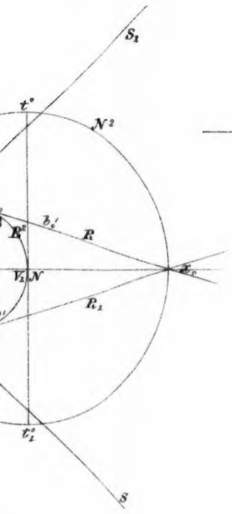


Fig. 3.

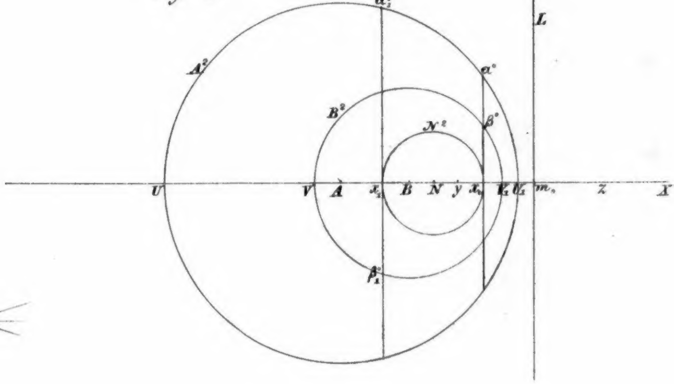
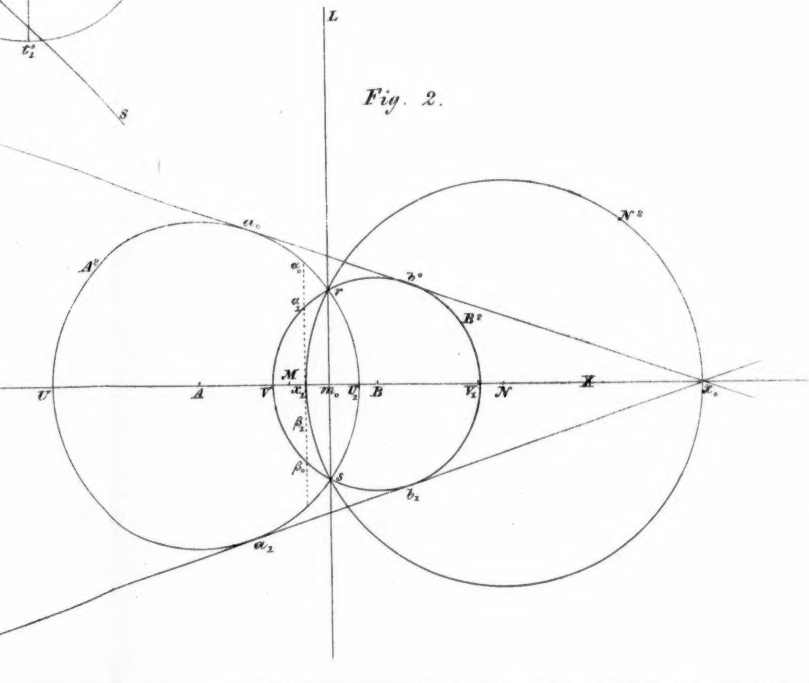


Fig. 2.



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